Dual-Ledger-Chrono-DM-B

Class-Limited Einstein Matches, a Closed Causal-Geodesic Obstruction, On-Shell Ordinary Shields, a Prescribed Misner Overlay, and One-Sided Sagnac Identifiability on Prescribed Hosts

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16 November 2026 (request t563u; typographic parent official Chrono-DM-B Version 4; version 5; wired=false)

Abstract

Official Dual-Ledger-Chrono-DM-A version 4 already recorded a complete flat cylindrical chronology-split witness, a printed switch kinetic action that closes the enlarged exchange ledger on the switch shell as action bookkeeping, a standard Gödel comparison compatible for dust plus a negative cosmological term together with an explicit canonical winding residual, a prescribed bimetric Misner overlay, and a host-null guided-ring identifiability statement that reconstructs (a, c, b) jointly from ordinary two-way arrivals together with a future host-null datum in each arrival class [1]. That official Chrono-DM-A record is cited only as a documentary source; every identity used below is re-proved locally. Official version 3 of the present companion re-proved every admitted identity from definitions printed there, classified the three Einstein-match branches on $g_{\alpha,\beta}$ that actually close, all with $\Lambda = -\beta^2/2$, typestated residuals honestly, and admitted a closed-timelike-geodesic obstruction, two typestated on-shell test-switch profiles, a Misner transmitting-versus-winding split, and a one-sided $c < 0$ Sagnac inversion. Those official version 3 identities remain in force. No official version 3 identity was found formally wrong.

This official version 5 is a typographic successor of official version 4. Official version 4 typestate, labels, and local proofs are retained in full. Authorised new work is typestate hygiene: ordinary quotation marks, a Scholar-parseable bibliography without an invented DOI, and a documentary OpenAI-family negative-control

table. Written dark-ledger identities remain identities on the written class; they are not discharges of the ordinary time-machine bar, not chronology engines in nature, and not uniqueness of dark matter. Clock Track-I identities are not imported as theorem-class. Official residuals on $g_{\alpha,\beta}$ are not withdrawn, and no winding-inclusive residual-free Einstein match is typestated beyond official version 4.

Official version 4 was a typestate-honest expansion of that spine. The documentary Chrono-DM-A citation is updated to official version 4, and the documentary Temporal-A citation is updated to official version 5. Local enhancements are admitted only after recomputation from the printed first integrals, switch equation, inverse metric, coframe, and Einstein tensor. On the unshifted family with $\alpha > 1$, there is no closed causal geodesic: the official timelike typestate is retained, including the bounce period $T_x = 2\pi/(|\beta|\sqrt{(\alpha^2 - 1)E^2 + 1})$ and $\Delta t > 0$, and every future-directed null bounce likewise has vanishing net azimuthal increment and a strictly positive, affine-invariant net time increment $\Delta t_{\text{null}} = \frac{2\pi}{|\beta|}(\alpha - \sqrt{\alpha^2 - 1})$. Closed non-geodesic causal curves already exist, and the statement is not geodesic completeness of the manifold. The official vacuum test-switch profiles are kept. Comoving ordinary dust drops out of the switch equation on the official overlay; a constant isotropic Jordan pressure can be absorbed into a shifted potential, which remains unbounded below for the linear switch, while a radially varying comoving pressure is not Jordan–Euler-on-shell. A second closed one-form $q_{(x)} = \kappa_x \, \mathrm{d}x$ on the unchanged printed metric does not yield an on-shell residual-free branch: current conservation and constant stiffness cannot be satisfied simultaneously, and the formal two-form stress sum is not the official Maxwell stress. On the prescribed Misner overlay, the switch wave operator and gradient square are computed from the official inverse metric; an affine switch has $(\nabla s)^2$ changing sign across $T = 0$, and the on-shell linear potential is unbounded below, while orthonormal and null components of the switch stress remain bounded across the horizon. A local vorticity comparison on the official exponential shield measures Δ_q by the ratio of ordinary to host precession scalars, and does not withdraw fibre-blindness of ordinary-only Sagnac data. Official residuals on $g_{\alpha,\beta}$ are not withdrawn, and Track II comparisons on other metric families, including the four named companion programmes of the t509u brief, are confined to the debugging appendix. No such companion is created.

The operational time-machine bar remains 0 of 3 as the full three-part bar. Host C is not created. O.1 stays NOT_CLOSED. Gate 0 and ST4b stay NOT_SUPPLIED. wired = false; declaredPath = null. No compilation or machine verification is asserted.

Bibliographic record

Title	Dual-Ledger-Chrono-DM-B
Author	Cheung, Ngo
Publication date	2026
Institution	Enter
Report number	02 Dual-Ledger-Chrono-DM-B-v5
Publisher	Cheung Ngo Medical Limited
First page	1
Last page	60
Language	en

Scope, typestate, and relation to official Chrono-DM-A

Why a second chronology sheet is needed

Official Dual-Ledger-Chrono-DM-A version 4 [1] *Documentary source record; not a theorem premise.* separated the informal claim that "dark matter travels in time while ordinary matter cannot" into backward signalling to ordinary apparatus (TM-1), bimetric chronology asymmetry on a prescribed host (TM-2), and register or echo non-identifications (TM-3), building on the disformal cone-nesting geometry of official Dual-Ledger-Mobility-A version 5 [2] *Documentary source record; not a theorem premise.* and the clock and transport discipline of official Dual-Ledger-Temporal-A version 5 [3] *Documentary source record; not a theorem premise.*. That official sheet already proved an exact TM-2 witness on a complete flat quotient $M = \mathbb{R}_t \times \mathbb{R}_x \times S^1_\varphi \times \mathbb{R}_z$, a closed-orbit Rolle obstruction, a printed switch kinetic action, a standard Gödel comparison with an explicit winding residual, a

prescribed Misner overlay, and a host-null Sagnac statement typestated by arrival class.

The attached Version 1 of the present companion treated four residual programmes as if official A had left them entirely open, and announced that they were resolved:

1. *Switch-shell bookkeeping.* When the disformal coupling $B(s)$ varies, an unsourced scalar switch leaves an un-enlarged remainder $\nabla_a T^{ab}_{\mathrm{tot}} = E_s \nabla^b s$. Official A already closed the enlarged four-slot identity on the switch shell as action bookkeeping, retaining that remainder if the kinetic slot is omitted.

2. *Einstein matching beyond the flat witness.* Because the flat quotient has $G_{ab} = 0$, positive conserved dark dust and canonical azimuthal winding stress cannot jointly solve $M_{\mathrm{P}}^2(G_{ab} + \Lambda g_{ab}) = T_{\mathrm{d},ab} + T_{\Theta,ab}$. Official A recorded that residual and a dust-plus-negative-Λ Gödel comparison whose winding residual likewise does not vanish. A later rotating-family match may be admitted only after a local recomputation, and only as a named source class.

3. *Entry from a shared causal past.* The stationary quotient carries closed host-timelike curves at all coordinate times $t \in \mathbb{R}$. A prescribed Misner overlay can exhibit a shared globally hyperbolic past and a dark Cauchy horizon; it does not manufacture that horizon from ordinary laboratory initial data, and it does not select a unique dark PDE continuation past $T = 0$.

4. *Sagnac fibre and host-null data.* Ideal ordinary guided-ring arrival times reconstruct only a and $H_r = c + b$. A declared host-null ring forward reduces the fibre where both future arrivals exist, and characterises the chronology-violating class by unidirectional azimuthal cone tilt.

Every theorem below is proved locally from definitions printed in this manuscript. Classical ingredients—disformal metrics [4], homogeneous rotating dust spacetimes [5, 6, 7], and Misner space [8, 9]—belong to the established general-relativity literature and are credited accordingly. Companion manuscripts are documentary source records, not premises.

What is and is not established

toprule Item Official typestate in this sheet

midrule Four-slot switch-shell conservation $\nabla_a(T_{\mathrm{o}}^{ab} + T_{\mathrm{d}}^{ab} + T_{\Theta}^{ab} + T_{s}^{ab}) = 0$ Re-proved locally as action bookkeeping (Theorem). Official A already printed the same enlargement. The un-enlarged remainder is retained if the kinetic slot is omitted.

Exact Riemann and Einstein tensors of $g_{\alpha,\beta}$ Proved via Cartan connection forms and confirmed by a coordinate-basis computation (Theorem).

Named Einstein-match classes on $g_{\alpha,\beta}$ Three residual-free branches are admitted after local algebra (Theorems –): dust plus negative Λ at $\alpha = \sqrt{2}$; dust plus a longitudinal closed one-form plus the same Λ for $\alpha \in (1, \sqrt{2}]$; force-free charged dust plus Maxwell plus the same Λ for $\alpha \in [\sqrt{2}, 2]$. Canonical azimuthal winding stress is not cancelled (Proposition). A second closed one-form $q_{(x)}$ on the unchanged printed metric does not yield an on-shell residual-free branch (Proposition). Official Branches G, L, and M, and $\Lambda = -\frac{1}{2}\beta^2$ on those branches, are not withdrawn. The Version 1 parameter set is withdrawn.

Closed host-timelike curves versus geodesics For $\alpha > 1$, every azimuthal circle is a closed future-directed host-timelike curve with acceleration βe_1, hence not a geodesic (Proposition). The unshifted family admits no closed causal geodesic (Theorem), including the official closed timelike obstruction and a closed-null classification recomputed from the printed first integrals. The statement is not geodesic completeness of the manifold, and is not a list of every closed causal curve. Circular geodesics are admitted only on the holonomy-shifted family at a special radius with $0 < \alpha < 1$ (Corollary).

Ordinary Cauchy foliation on the rotating host Proved on the region where the printed growth condition on $H(x)$ holds (Theorem). A constant weight on a half-space is not a complete ordinary-Cauchy witness on all of M. The exponential profile is realised by two typestated on-shell test-switch solutions of with $\mathscr{T}^{ab} = 0$ (Lemma), not as Einstein back-reaction. Comoving ordinary dust drops out of the switch equation; a constant isotropic Jordan pressure is absorbed into a shifted potential, while a radially varying comoving pressure is not Jordan–Euler-on-shell (Proposition). Neither source is Einstein back-reaction on $g_{\alpha,\beta}$.

Prescribed Misner overlay Shared globally hyperbolic past, dark Cauchy horizon at $T = 0$, and ordinary global

hyperbolicity wherever $H_M(T) > 0$ (Theorem). A prescribed ramp is not dynamical Einstein manufacture and is not Bar 3. An on-shell potential audit of $s = s(T)$ is admitted after recomputing $\mathrm{Box}_{g_M} s$ and $(\nabla s)^2$ from the official inverse metric (Proposition); it does not withdraw the prescribed typestate. Future-directed causal geodesics and their Vlasov lift split into a transmitting class $J < 0$ and an infinitely winding class $J \geq 0$ (Proposition).

Dark Klein–Gordon modes across $T = 0$ The separated Fuchsian analysis is admitted (Proposition). It does not select a unique dark PDE continuation past $T = 0$.

Ordinary plus host-null Sagnac identifiability Joint two-future-time reconstruction of (a, c, b) is used only where both future host-null arrivals exist. In the class $c < 0$, ordinary two-way arrivals together with the single existing future host-null arrival $t_-^{(g)} > 0$ reconstruct (a, c, b) algebraically (Theorem). The negative-$t_+^{(g)}$ product remains unused. The chronology-violating class remains typestated by unidirectional azimuthal cone tilt. A local vorticity comparison on the official exponential shield measures Δ_q (Proposition), and does not withdraw fibre-blindness of ordinary-only Sagnac data, nor replace Theorem or Proposition .

Laboratory manufacture of CTCs, operational time-machine closure, Host C, or a quantum-gravity completion Refused. The inherited time-machine score stays 0 of 3.

bottomrule

Standing bars

Host A and Host B stay separate prescribed backgrounds. Host C is not created. O.1 stays NOT_CLOSED. XV-B is not opened. XVI-N and Route B are not discharged except the class-limited negative of t441 XVI-N-Univ-A. Gate 0 and ST4b stay NOT_SUPPLIED. No quantum coefficient is admitted. No derived metric, spacetime emergence, Page curve, island, bounce, or vacuum-energy claim is made. H3, A1–A3, and the phenomenal-theory gate remain NOT_CLOSED and unused. The operational time-machine bar remains 0 of 3 as the full three-part bar. Exhibiting a named Einstein-compatible background, an on-shell test-switch profile, a closed causal-geodesic obstruction, a Misner switch-stress audit, a local gyroscope comparison, or a switch-shielded horizon crossing on a

prescribed manifold, does not manufacture a laboratory time machine from ordinary matter, select a unique dark PDE continuation past $T = 0$, or remove external geometric time. A Track II appendix comparison on a different metric family does not withdraw official residuals on $g_{\alpha,\beta}$, and is not Bar 3. An Einstein match on a printed ansatz is host-compatibility bookkeeping. It is not Einstein dynamics, not a derived metric, and not Host C. Companion records and the OpenAI mathematics catalogue (supplied documentary record dated 7 October 2026) are source records only, not premises. wired = false; declaredPath = null. No compilation or machine verification is asserted.

Four-slot switch-shell bookkeeping

We use signature $(-,+,+,+)$, units $c = 1$, and reduced gravitational constant $M_{\mathrm{p}} := (8\pi G)^{-1/2} > 0$. Let (M, g) be a smooth time-oriented four-dimensional Lorentzian manifold with volume form $\mathrm{dvol}_g = \sqrt{-g}\,\mathrm{d}^4x$ and Levi-Civita connection ∇. Hilbert stress tensors are defined by $T_X^{ab} := \frac{2}{\sqrt{-g}}\frac{\delta S_X}{\delta g_{ab}}$.

The following class is the same enlargement already used in official Chrono-DM-A version 4 [1]: a shift-symmetric winding controller together with a canonical switch kinetic action. The identities of this section are re-proved locally. They are action bookkeeping on a prescribed host, not Einstein dynamics and not a claim that official A left the switch ledger completely open.

Definition (Dynamically switched winding class $\mathfrak{W}_{\mathrm{dyn}}$). Fix a reference closed one-form q_0 on M ($\mathrm{d}q_0 = 0$). An admissible configuration consists of:

1. a closed winding one-form $q = q_0 + \mathrm{d}\eta$ with $\eta \in C^\infty(M;\mathbb{R})$ (fixing the de Rham class $[q] = [q_0] \in H^1(M;\mathbb{R})$ under compactly supported variations $\delta q = \mathrm{d}(\delta\eta)$),

2. a real switch scalar $s \in C^\infty(M;\mathbb{R})$ with kinetic scale $F_s > 0$, smooth potential $W \in C^\infty(\mathbb{R};\mathbb{R})$, and smooth non-negative disformal coupling $B \in C^\infty(\mathbb{R};[0,\infty))$,

3. an ordinary-sector metric

$$\hat{g}_{ab} := g_{ab} + B(s)\,q_a q_b, \qquad \sigma := g^{ab} q_a q_b, \qquad \Delta_q := 1 + B(s)\sigma > 0,$$

4. a four-sector local action functional on regular relatively compact domains $U \mathrm{Subset} M$:

$$S_U = S_{\mathrm{o},U}[\hat{g}, \psi_{\mathrm{o}}, \mathscr{A}] + S_{\mathrm{d},U}[g, \psi_{\mathrm{d}}] + S_{\Theta,U}[g, q] + S_{s,U}[g, s],$$

where $S_{\mathrm{d},U}$ is independent of $(\mathscr{A}, q, s, \psi_{\mathrm{o}})$ (exact off-shell electromagnetic and controller silence), and the controller and switch actions are

$$S_{\Theta,U}[g, q] = -\frac{M_\Theta^2}{2}\int_U g^{ab} q_a q_b \,\mathrm{dvol}_g, \qquad S_{s,U}[g, s] = -\int_U \left(\frac{F_s^2}{2} g^{ab} \nabla_a s \nabla_b s + W(s)\right) \mathrm{dvol}_g,$$

with constant stiffness scale $M_\Theta > 0$.

Lemma (Lorentzian inverse and causal cone containment). *Wherever $\Delta_q = 1 + B(s)\sigma > 0$ and $B(s) \geq 0$, the tensor $\hat{g}_{ab}$ has Lorentzian signature $(-,+,+,+)$, volume ratio $\sqrt{-\hat{g}}/\sqrt{-g} = \sqrt{\Delta_q}$, and exact inverse*

$$\hat{g}^{ab} = g^{ab} - \frac{B(s)}{\Delta_q} q^a q^b \qquad (q^a := g^{ac} q_c).$$

For every tangent vector v^a, $g_{ab} v^a v^b = \hat{g}_{ab} v^a v^b - B(s)(q_a v^a)^2$, so every $\hat{g}$-causal vector is g-causal, and when time orientations are aligned on a common timelike field, $J^+_{\hat{g}}(p) \subseteq J^+_g(p)$.

Proof. Contracting with yields $\hat{g}^{ab}\hat{g}_{bc} = \delta^a_c + B(1 - \frac{1+B\sigma}{\Delta_q}) q^a q_c = \delta^a_c$. By the matrix determinant lemma, $\det(\hat{g}_{ab}) = \Delta_q \det(g_{ab}) < 0$. Along the homotopy $\lambda \mapsto g + \lambda B(s) q \otimes q$, $\lambda \in [0,1]$, the determinant is $(1 + \lambda B(s)\sigma)\det g$ with $1 + \lambda B(s)\sigma \geq \min(1, \Delta_q) > 0$, so the signature $(-,+,+,+)$ is preserved. The causal inequality follows from $B(s)(q_a v^a)^2 \geq 0$.

Theorem (Four-slot switch-shell conservation, as action bookkeeping). *Let $\hat{T}^{ab}_{\mathrm{o}} := \frac{2}{\sqrt{-\hat{g}}} \frac{\delta S_{\mathrm{o}}}{\delta \hat{g}_{ab}}$ denote the Jordan-frame ordinary Hilbert stress, and set $\mathscr{T}^{ab} := \sqrt{\Delta_q}\, \hat{T}^{ab}_{\mathrm{o}}$. For any configuration in $\mathfrak{W}_{\mathrm{dyn}}$:*

***1.** The four g-frame Hilbert stress tensors are $T^{ab}_{\mathrm{o}} = \mathscr{T}^{ab}$, T^{ab}_{d}, and*

$$T^{ab}_\Theta = M_\Theta^2\left(q^a q^b - \frac{1}{2} g^{ab}\sigma\right), \qquad T^{ab}_s = F_s^2 \nabla^a s \nabla^b s - g^{ab}\left(\frac{F_s^2}{2}(\nabla s)^2 + W(s)\right).$$

2. *The Euler–Lagrange equations for η (within the fixed class $q = q_0 + \mathrm{d}\eta$) and for s are*

$$\nabla_a(M_\Theta^2 q^a - B(s)\mathscr{I}^{ab}q_b) = 0,$$

$$F_s^2 \mathrm{Box}_g\, s - W'(s) + \frac{1}{2}B'(s)\mathscr{I}^{ab}q_a q_b = 0.$$

3. *Whenever ψ_{o} and ψ_{d} satisfy their matter equations of motion, define $Q_\Theta := -\nabla_a(B(s)\mathscr{I}^{ab}q_b)$ and $E_s := \frac{1}{2}B'(s)\mathscr{I}^{ab}q_a q_b$. Then $\nabla_a T_{\mathrm{d}}^{ab} = 0$ and*

$$\nabla_a T_{\mathrm{o}}^{ab} = Q_\Theta q^b + E_s \nabla^b s, \qquad \nabla_a T_\Theta^{ab} = (M_\Theta^2 \nabla_a q^a)q^b, \qquad \nabla_a T_s^{ab} = (F_s^2 \mathrm{Box}_g\, s - W'(s))\nabla^b s.$$

When and also hold, both exchange terms cancel pointwise and the total four-slot stress is identically conserved:

$$\nabla_a(T_{\mathrm{o}}^{ab} + T_{\mathrm{d}}^{ab} + T_\Theta^{ab} + T_s^{ab}) = 0, \qquad \nabla_a(T_{\mathrm{o}}^{ab} + T_\Theta^{ab} + T_s^{ab}) = 0.$$

If the independent kinetic slot $S_{s,U}$ is omitted, the un-enlarged remainder $\nabla_a(T_{\mathrm{o}}^{ab} + T_{\mathrm{d}}^{ab} + T_\Theta^{ab}) = E_s\nabla^b s$ is recovered.

4. *If $W(s) \geq 0$ and $\widehat{T}_{\mathrm{o}}^{ab}$ satisfies the $\widetilde{g}$-weak energy condition ($\widehat{T}_{\mathrm{o},ab}w^a w^b \geq 0$ for all $\widetilde{g}$-causal w^a), while $T_{\mathrm{d}}^{ab} = \rho_{\mathrm{d}} u_{\mathrm{d}}^a u_{\mathrm{d}}^b$ with $\rho_{\mathrm{d}} \geq 0$, then all four slots individually satisfy the g-null energy condition: $T_X^{ab}k_a k_b \geq 0$ for every g-null covector k_a.*

Proof. (a) At fixed (q, s), one has $\delta\widetilde{g}_{ab} = \delta g_{ab}$, so $\sqrt{-g}\, T_{\mathrm{o}}^{ab} = \sqrt{-\widetilde{g}}\,\widehat{T}_{\mathrm{o}}^{ab} = \sqrt{-g}\,\mathscr{I}^{ab}$. Direct metric variation of gives .

(b) Varying $\eta \mapsto \eta + \delta\eta$ with $\mathrm{supp}\,\delta\eta \mathrm{Subset}\, U$ gives $\delta\widetilde{g}_{ab} = 2B(s)q_{(a}\nabla_{b)}\delta\eta$, so integrating by parts in $\delta_\eta(S_{\mathrm{o},U} + S_{\Theta,U}) = \int_U (B(s)\mathscr{I}^{ab}q_b - M_\Theta^2 q^a)\nabla_a\delta\eta\,\mathrm{dvol}_g$ yields . Varying $s \mapsto s + \delta s$ gives $\delta_s\widetilde{g}_{ab} = B'(s)q_a q_b\delta s$, so $\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{o},U}}{\delta s} = \frac{1}{2}B'(s)\mathscr{I}^{ab}q_a q_b = E_s$, and adding $\frac{1}{\sqrt{-g}}\frac{\delta S_{s,U}}{\delta s} = F_s^2\mathrm{Box}_g\, s - W'(s)$ yields .

(c) For any $\xi \in C_c^\infty(U; TM)$, Cartan's magic formula with $\mathrm{d}q = 0$ gives $\mathscr{L}_\xi q = \mathrm{d}(\iota_\xi q) + \iota_\xi \mathrm{d}q = \mathrm{d}(\xi^c q_c)$, which is an exact variation $\delta\eta = \xi^c q_c$, alongside $\mathscr{L}_\xi s = \xi^c\nabla_c s$ and $\mathscr{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}$. Diffeomorphism invariance of $S_{\mathrm{o},U}$ on the ψ_{o} -shell gives

$$0 = \int_U (T_{\mathrm{o}}^{ab}\nabla_a\xi_b + Q_\Theta(\xi^b q_b) + E_s(\xi^b\nabla_b s))\mathrm{dvol}_g = \int_U (-\nabla_a T_{\mathrm{o}}^{ab} + Q_\Theta q^b + E_s\nabla^b s)\xi_b\,\mathrm{dvol}_g,$$

proving the first identity in ; the same argument on $S_{\mathrm{d},U}$ gives $\nabla_a T^{ab}_{\mathrm{d}} = 0$. Because $\mathrm{d}q = 0$ implies $\nabla_a q_b = \nabla_b q_a$, differentiating cancels $q^a(\nabla_a q^b - \nabla^b q_a) = 0$ and gives $\nabla_a T^{ab}_{\Theta} = M^2_{\Theta}(\nabla_a q^a)q^b$. Differentiating T^{ab}_s, using $\nabla_a \nabla^b s = \nabla^b \nabla_a s$ to cancel second derivatives, gives the third identity in . Substituting ($Q_{\Theta} + M^2_{\Theta}\nabla_a q^a = 0$) and ($E_s + F^2_s \mathrm{Box}_g s - W'(s) = 0$) produces . Omitting $S_{s,U}$ drops T^{ab}_s and leaves the remainder $E_s \nabla^b s$.

(d) For g-null k_a, one has $g^{ab}k_a k_b = 0$, so $T^{ab}_{\Theta}k_a k_b = M^2_{\Theta}(q^a k_a)^2 \geq 0$, $T^{ab}_s k_a k_b = F^2_s(k_a \nabla^a s)^2 \geq 0$, and $T^{ab}_{\mathrm{d}}k_a k_b = \rho_{\mathrm{d}}(u^a_{\mathrm{d}}k_a)^2 \geq 0$. By , $\tilde{g}^{ab}k_a k_b = -\frac{B(s)}{\Delta_q}(q^a k_a)^2 \leq 0$, so $w^a := \tilde{g}^{ab}k_b$ is $\tilde{g}$-causal, whence $T^{ab}_{\mathrm{o}}k_a k_b = \sqrt{\Delta_q}\,\widehat{T}_{\mathrm{o},ab}w^a w^b \geq 0$.

Remark (Relation to official Chrono-DM-A). Theorem is a local re-proof of the enlarged switch-shell identity already stated as official Chrono-DM-A Theorem thm:switchkin [1]. It does not withdraw that official identity, and it does not claim that official A left the switch ledger completely open. The identity remains action bookkeeping on a prescribed host: it does not derive the Einstein equation, does not manufacture a switch profile, and is not Host C.

The rotating family: curvature, closed curves, and geodesics

The flat quotient of official Chrono-DM-A has vanishing Einstein tensor, so it cannot match positive conserved dark dust together with canonical winding stress [1]. That flat residual is not withdrawn by anything in this section. We analyse a two-parameter family of stationary, cylindrically periodic, homogeneous rotating metrics on

$$M = \mathbb{R}_t \times \mathbb{R}_x \times S^1_{\varphi} \times \mathbb{R}_z, \qquad \varphi \sim \varphi + 2\pi,$$

which encompasses Gödel-type homogeneous rotating universes [5, 7]. The Som–Raychaudhuri charged-dust cylinder [6] is a named comparison only: it is inhomogeneous in the cylindrical radius, and is not identical with any member of the homogeneous family printed below.

Metric, coframe, and Einstein tensor

Fix real constants $\alpha > 0$ and $\beta \neq 0$, and prescribe on M the smooth metric

$$g_{\alpha,\beta} := -(\mathrm{d}t + \alpha e^{\beta x}\mathrm{d}\varphi)^2 + \mathrm{d}x^2 + e^{2\beta x}\mathrm{d}\varphi^2 + \mathrm{d}z^2 = -\mathrm{d}t^2 - 2\alpha e^{\beta x}\mathrm{d}t\,\mathrm{d}\varphi + (1-\alpha^2)e^{2\beta x}\mathrm{d}\varphi^2 + \mathrm{d}x^2 + \mathrm{d}z^2.$$

Introduce the globally defined orthonormal coframe $(\theta^0, \theta^1, \theta^2, \theta^3)$, in which $g_{\alpha,\beta} = -(\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2$:

$$\theta^0 := \mathrm{d}t + \alpha e^{\beta x}\mathrm{d}\varphi, \qquad \theta^1 := \mathrm{d}x, \qquad \theta^2 := e^{\beta x}\mathrm{d}\varphi, \qquad \theta^3 := \mathrm{d}z,$$

with dual orthonormal vector frame

$$e_0 = \partial_t, \qquad e_1 = \partial_x, \qquad e_2 = e^{-\beta x}\partial_\varphi - \alpha\partial_t, \qquad e_3 = \partial_z.$$

Align the time orientation of $(M, g_{\alpha,\beta})$ with the unit timelike Killing vector $u_{\mathrm{d}} := e_0 = \partial_t$ ($\theta^0(u_{\mathrm{d}}) = +1$). The only nonzero inverse components are

$$g^{tt} = \alpha^2 - 1, \qquad g^{t\varphi} = -\alpha e^{-\beta x}, \qquad g^{\varphi\varphi} = e^{-2\beta x}, \qquad g^{xx} = g^{zz} = 1,$$

and $\det g_{\alpha,\beta} = -e^{2\beta x}$.

Theorem (Exact curvature and Einstein tensor of $g_{\alpha,\beta}$). *For every $\alpha > 0$ and $\beta \neq 0$ on $(M, g_{\alpha,\beta})$:*

1. *The spacetime is smooth, Lorentzian, and time-orientable. On the universal cover $\widetilde{M} = \mathbb{R}^4$ the metric is geodesically complete, and the quotient by the free isometry $\varphi \mapsto \varphi + 2\pi$ is likewise geodesically complete.*

2. *In the orthonormal coframe , the non-zero Einstein tensor components $G_{\hat{a}\hat{b}} := G(e_{\hat{a}}, e_{\hat{b}})$ are constant on M:*

$$G_{\hat{0}\hat{0}} = \frac{\beta^2}{4}(3\alpha^2 - 4), \qquad G_{\hat{1}\hat{1}} = G_{\hat{2}\hat{2}} = \frac{\beta^2\alpha^2}{4}, \qquad G_{\hat{3}\hat{3}} = \frac{\beta^2}{4}(4 - \alpha^2),$$

with all off-diagonal components vanishing. The orthonormal Ricci tensor and scalar curvature are

$$R_{\hat{0}\hat{0}} = \frac{\alpha^2\beta^2}{2}, \qquad R_{\hat{1}\hat{1}} = R_{\hat{2}\hat{2}} = \frac{\beta^2}{2}(\alpha^2 - 2), \qquad R_{\hat{3}\hat{3}} = 0, \qquad R = \frac{\beta^2}{2}(\alpha^2 - 4).$$

3. *The dark fluid velocity $u_{\mathrm{d}} = \partial_t$ is geodesic and expansion-free: $\nabla_{u_{\mathrm{d}}} u_{\mathrm{d}} = 0$ and $\nabla_a u_{\mathrm{d}}^a = 0$. Thus for any constant ρ_{d}, the dust stress $T_{\mathrm{d}}^{ab} := \rho_{\mathrm{d}} u_{\mathrm{d}}^a u_{\mathrm{d}}^b$ satisfies $\nabla_a T_{\mathrm{d}}^{ab} = 0$ identically.*

Proof. (a) Smoothness, Lorentzian signature, and time-orientability are immediate from –. On the cover, ∂_t, ∂_φ,

and ∂_z are commuting Killing fields. Along any geodesic $\gamma(s)$ the conserved momenta $E := -g(\partial_t, \dot\gamma) = \dot t + \alpha e^{\beta x}\dot\varphi$, $J := g(\partial_\varphi, \dot\gamma) = -\alpha e^{\beta x} E + e^{2\beta x}\dot\varphi$, and $P_z := \dot z$ are constant. The normalisation $g(\dot\gamma, \dot\gamma) = \varepsilon \in \{-1, 0, +1\}$ then becomes

$$\dot x^2 + (J e^{-\beta x} + \alpha E)^2 + P_z^2 - E^2 = \varepsilon,$$

hence $\dot x^2 \le E^2 + |\varepsilon|$. The bound on $|\dot x|$ prevents blow-up of $x(s)$ in finite affine time, so $e^{\pm\beta x}$, $\dot\varphi$, $\dot t$, and $\dot z$ remain smooth on every bounded affine interval. The cover is therefore geodesically complete, and so is the quotient by the fixed-point-free isometry generated by ∂_φ.

(b) The exterior derivatives of are

$$\mathrm{d}\theta^0 = \alpha\beta\,\theta^1 \wedge \theta^2, \qquad \mathrm{d}\theta^1 = 0, \qquad \mathrm{d}\theta^2 = \beta\,\theta^1 \wedge \theta^2, \qquad \mathrm{d}\theta^3 = 0.$$

The unique metric-compatible, torsion-free connection forms satisfying $\mathrm{d}\theta^{\hat a} + \omega^{\hat a}{}_{\hat b} \wedge \theta^{\hat b} = 0$ and $\omega_{\hat a\hat b} = -\omega_{\hat b\hat a}$ are

$$\omega^{\hat 0}{}_{\hat 1} = \omega^{\hat 1}{}_{\hat 0} = -\frac{\alpha\beta}{2}\theta^2, \qquad \omega^{\hat 0}{}_{\hat 2} = \omega^{\hat 2}{}_{\hat 0} = \frac{\alpha\beta}{2}\theta^1, \qquad \omega^{\hat 1}{}_{\hat 2} = -\omega^{\hat 2}{}_{\hat 1} = -\beta\theta^2 + \frac{\alpha\beta}{2}\theta^0,$$

with all components involving the index $\hat 3$ vanishing. Torsion-freeness is the direct check

$$-\omega^{\hat 0}{}_{\hat 1} \wedge \theta^1 - \omega^{\hat 0}{}_{\hat 2} \wedge \theta^2 = -\alpha\beta\,\theta^1 \wedge \theta^2 = -\mathrm{d}\theta^0,$$

$$-\omega^{\hat 1}{}_{\hat 0} \wedge \theta^0 - \omega^{\hat 1}{}_{\hat 2} \wedge \theta^2 = 0 = -\mathrm{d}\theta^1,$$

$$-\omega^{\hat 2}{}_{\hat 0} \wedge \theta^0 - \omega^{\hat 2}{}_{\hat 1} \wedge \theta^1 = -\beta\,\theta^1 \wedge \theta^2 = -\mathrm{d}\theta^2.$$

The curvature two-forms $\Omega^{\hat a}{}_{\hat b} = \mathrm{d}\omega^{\hat a}{}_{\hat b} + \omega^{\hat a}{}_{\hat c} \wedge \omega^{\hat c}{}_{\hat b}$ are then

$$\Omega^{\hat 0}{}_{\hat 1} = \frac{\alpha^2\beta^2}{4}\,\theta^0 \wedge \theta^1,$$

$$\Omega^{\hat 0}{}_{\hat 2} = \frac{\alpha^2\beta^2}{4}\,\theta^0 \wedge \theta^2,$$

$$\Omega^{\hat 1}{}_{\hat 2} = \frac{\beta^2}{4}(3\alpha^2 - 4)\,\theta^1 \wedge \theta^2,$$

and all two-forms with a $\hat{3}$ index vanish. (The arithmetic of is

$-\beta\,\mathrm{d}\theta^2 + \frac{\alpha\beta}{2}\mathrm{d}\theta^0 + \omega^{\hat{1}}{}_{\hat{0}} \wedge \omega^{\hat{0}}{}_{\hat{2}} = (-\beta^2 + \frac{1}{2}\alpha^2\beta^2 + \frac{1}{4}\alpha^2\beta^2)\theta^1 \wedge \theta^2.$)

To convert these two-forms into the Einstein tensor used in the field equation, we compute in the coordinate basis, with the standard conventions $\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$, $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$, and $R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu}$. The only nonzero Christoffel symbols, up to symmetry in the lower pair, are

$$\Gamma^{t}{}_{tx} = \frac{1}{2}\alpha^2\beta, \qquad \Gamma^{t}{}_{x\varphi} = \frac{1}{2}\alpha\beta(\alpha^2 - 1)e^{\beta x}, \ \Gamma^{x}{}_{t\varphi} = \frac{1}{2}\alpha\beta e^{\beta x},$$

$$\Gamma^{x}{}_{\varphi\varphi} = \beta(\alpha^2 - 1)e^{2\beta x}, \ \Gamma^{\varphi}{}_{tx} = -\frac{1}{2}\alpha\beta e^{-\beta x}, \qquad \Gamma^{\varphi}{}_{x\varphi} = \frac{1}{2}\beta(2 - \alpha^2).$$

Substitution yields the coordinate Ricci components

$$R_{tt} = \frac{1}{2}\alpha^2\beta^2, \qquad R_{t\varphi} = \frac{1}{2}\alpha^3\beta^2 e^{\beta x}, \qquad R_{xx} = \frac{1}{2}\beta^2(\alpha^2 - 2), \qquad R_{\varphi\varphi} = \frac{1}{2}\beta^2(\alpha^4 + \alpha^2 - 2)e^{2\beta x},$$

with $R_{zz} = 0$ and scalar curvature $R = \frac{1}{2}\beta^2(\alpha^2 - 4)$. The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is therefore

$$G_{tt} = \frac{\beta^2}{4}(3\alpha^2 - 4), \qquad G_{t\varphi} = \frac{\alpha\beta^2}{4}(3\alpha^2 - 4)e^{\beta x}, \ G_{xx} = \frac{\alpha^2\beta^2}{4},$$

$$G_{\varphi\varphi} = \frac{3\alpha^2\beta^2}{4}(\alpha^2 - 1)e^{2\beta x}, \ G_{zz} = \frac{\beta^2}{4}(4 - \alpha^2).$$

Evaluating $G_{\hat{a}\hat{b}} = G_{\mu\nu}e_{\hat{a}}^{\mu}e_{\hat{b}}^{\nu}$ on the frame produces , and the same transcription of produces . In particular, at the Gödel value $\alpha = \sqrt{2}$ one has $G_{\hat{a}\hat{b}} = \frac{1}{2}\beta^2\,\mathrm{diag}\,(+1, +1, +1, +1)$ and $R_{\hat{a}\hat{b}} = \beta^2\delta_{\hat{a}0}\delta_{\hat{b}0}$, which is the standard homogeneous dust-plus-negative-Λ typestate recalled in Section .

(c) From , $\omega^{\hat{a}}{}_{\hat{0}}(e_0) = 0$ for all $\hat{a}$, so $\nabla_{e_0}e_0 = 0$. The expansion vanishes because $\nabla_a u^a_{\mathrm{d}} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\delta^{\mu}_{t}) = 0$.

Closed timelike curves are not automatically geodesics

Proposition (Azimuthal circles: closed timelike curves, not geodesics). *Let $\alpha > 1$ and $\beta \neq 0$. For every (t_0, x_0, z_0), the azimuthal circle $\gamma_{t_0,x_0,z_0}(\varphi) = (t_0, x_0, \varphi, z_0)$, $\varphi \in [0, 2\pi]$, is a smooth closed future-directed $g_{\alpha,\beta}$-timelike curve, with $g_{\alpha,\beta}(\partial_\varphi, \partial_\varphi) = (1 - \alpha^2)e^{2\beta x_0} < 0$ and $\theta^0(\partial_\varphi) = \alpha e^{\beta x_0} > 0$. Its unit tangent $v = \frac{e^{-\beta x_0}}{\sqrt{\alpha^2 - 1}}\partial_\varphi = \frac{1}{\sqrt{\alpha^2 - 1}}(\alpha e_0 + e_2)$ has*

proper period $L(x_0) = 2\pi\sqrt{\alpha^2 - 1}\, e^{\beta x_0}$ *and covariant acceleration*

$$\nabla_v v = \beta\, e_1$$

at every $\alpha > 1$. *In particular, these circles are not geodesics unless* $\beta = 0$, *which is excluded. The Version 1 assertion that every such circle is a geodesic at* $\alpha = 2$, *and the Version 1 formula* $\nabla_v v = \frac{\beta(\alpha^2 - 2)}{\alpha^2 - 1} e_1$, *are both withdrawn.*

Proof. The causal character and future orientation are the evaluations just stated. The only nonzero contribution to the coordinate acceleration of the proper-time parameterisation $\dot{\varphi} = e^{-\beta x_0}/\sqrt{\alpha^2 - 1}$ is

$$(\nabla_v v)^x = \Gamma^x{}_{\varphi\varphi}\dot{\varphi}^2 = \beta(\alpha^2 - 1)e^{2\beta x_0} \cdot \frac{e^{-2\beta x_0}}{\alpha^2 - 1} = \beta,$$

with all other components vanishing, which is . Equivalently, for the Killing field $K = \partial_\varphi$ one has $\nabla_K K = -\frac{1}{2}\mathrm{grad}\,(K \cdot K)$ and $K \cdot K = (1 - \alpha^2)e^{2\beta x}$, so $(\nabla_K K)^x = \beta(\alpha^2 - 1)e^{2\beta x}$; dividing by the constant squared length $\lambda^2 = (\alpha^2 - 1)e^{2\beta x}$ along the circle again yields βe_1. The right-hand side is independent of α and vanishes only if $\beta = 0$.

Remark (Gödel closed curves versus closed geodesics). The distinction is classical: in Gödel's spacetime there are closed timelike curves through every event, while closed timelike geodesics require a quotient or a special construction [5, 9]. Proposition is compatible with that typestate. At $\alpha = \sqrt{2}$ the azimuthal circles are closed future-directed timelike curves with constant proper acceleration βe_1, not geodesics. Official version 2 left open the existence or non-existence of non-azimuthal closed timelike geodesics on the printed quotient. Official version 3 closed that question for the unshifted family at $\alpha > 1$, leaving closed null geodesics unclassified. Theorem now classifies the null class as well. The holonomy-shifted class of Corollary is not withdrawn, and no *t*-quotient is introduced.

Theorem (Closed causal-geodesic obstruction on the unshifted family). *Let* $\alpha > 1$ *and* $\beta \neq 0$. *The unshifted spacetime* $(M, g_{\alpha,\beta})$ *of – admits no closed causal geodesic: it admits no closed timelike geodesic and no closed null*

geodesic. More precisely, along any $g_{\alpha,\beta}$-geodesic the momenta $E = \dot{t} + \alpha e^{\beta x}\dot{\varphi}$, $J = -\alpha e^{\beta x}E + e^{2\beta x}\dot{\varphi}$, and $P_z = \dot{z}$ of Theorem (a) are constant, and a unit-timelike parameterisation obeys the radial identity

$$\dot{x}^2 + (Je^{-\beta x} + \alpha E)^2 = E^2 - P_z^2 - 1 =: K.$$

Writing $w := Je^{-\beta x} + \alpha E$, the remaining velocities are

$$\dot{t} = E - \alpha w, \qquad \dot{\varphi} = Je^{-2\beta x} + \alpha E e^{-\beta x}.$$

A closed curve in M requires a return of x, a vanishing net increment of z, a net increment of φ in $2\pi\mathbb{Z}$, and a vanishing net increment of t. Then:

1. *Azimuthal circles are not geodesics (Proposition).*

2. *Every closed candidate must have $P_z = 0$.*

3. *The only constant-x unit-timelike geodesics occur at the equilibrium $Je^{-\beta x_0} = -\alpha E$ with $K = 0$, where $\dot{\varphi} = 0$ and $\dot{t} = E = \pm 1$. In particular $\Delta t \neq 0$ on any positive proper-time interval, so these equilibria do not close.*

4. *Every future-directed unit-timelike geodesic with $E > 0$ and $P_z = 0$ has $J < 0$ (for $\alpha > 0$) and $K = E^2 - 1 \geq 0$. If $K = 0$, the orbit is an equilibrium of the type in (c). If $K > 0$, then $x(\tau)$ is periodic with proper period*

$$T_x = \frac{2\pi}{|\beta|\sqrt{(\alpha^2 - 1)E^2 + 1}},$$

the net azimuthal increment over each period vanishes, and the net time increment is

$$\Delta t = \frac{2\pi}{|\beta|}\left(\alpha + \frac{(1 - \alpha^2)E}{\sqrt{(\alpha^2 - 1)E^2 + 1}}\right) > 0.$$

Hence the orbit does not close in t.

5. *If $P_z \neq 0$, then $\Delta z = P_z \Delta\tau \neq 0$ on any positive proper-time interval, so the orbit does not close in z. In particular, for $\alpha > 1$ a runaway (non-periodic) radial motion would require $K \geq \alpha^2 E^2$, which already forces $P_z \neq 0$. The official unit-timelike typestate, including the bounce period , is not withdrawn. For a null geodesic, the affine first*

integral of Theorem (a) at $\varepsilon = 0$ *is*

$$\dot{x}^2 + (Je^{-\beta x} + \alpha E)^2 = E^2 - P_z^2 =: K_0,$$

and the velocities remain valid, now with respect to an affine parameter. Then:

1. *If* $E = 0$*, then* $K_0 = -P_z^2 \le 0$*, which forces* $P_z = 0$*,* $\dot{x} = 0$*, and* $J = 0$*, hence the zero tangent. There is no nondegenerate null geodesic in this class.*

2. *The only constant-x null geodesics occur at the equilibrium* $Je^{-\beta x_0} = -\alpha E$ *with* $K_0 = 0$*, hence* $P_z = \pm E$*, where* $\dot{\varphi} = 0$ *and* $\dot{t} = E$*. If* $E \neq 0$*, then both* Δt *and* Δz *are nonzero on any positive affine interval, so these equilibria do not close.*

3. *Every future-directed null geodesic with* $E > 0$ *and* $P_z = 0$ *has* $J < 0$ *(for* $\alpha > 0$*) and* $K_0 = E^2 > 0$*. Then x is periodic with affine period*

$$T_{x,\mathrm{null}} = \frac{2\pi}{|\beta| E \sqrt{\alpha^2 - 1}},$$

the net azimuthal increment over each period vanishes, and the net time increment is

$$\Delta t_{\mathrm{null}} = \frac{2\pi}{|\beta|}(\alpha - \sqrt{\alpha^2 - 1}) > 0 \qquad (\forall\, \alpha > 1).$$

The increment is independent of the affine scale E. Hence the orbit does not close in t.

4. *If* $P_z \neq 0$*, then* $\Delta z = P_z \Delta\lambda \neq 0$ *on any positive affine interval, so the orbit does not close in z. For* $\alpha > 1$ *there is in fact no runaway (non-periodic) null radial motion:* $K_0 = E^2 - P_z^2 \le E^2 < \alpha^2 E^2$*. The statement is a closed causal-geodesic obstruction on the unshifted printed family. It is not geodesic completeness of the manifold (already recorded in Theorem (a)), it is not a classification of every closed causal curve (closed non-geodesic host-timelike curves already exist by Proposition), and it is not a claim that the spacetime is causally simple. It does not withdraw Corollary on the holonomy-shifted family, it does not introduce a t-quotient, and it is not Bar 3.*

Proof. A smooth closed curve in $M = \mathbb{R}_t \times \mathbb{R}_x \times S^1_\varphi \times \mathbb{R}_z$ must return to the same (t, x, z) and to the same φ modulo

2π. Along a geodesic, P_z is constant, so $\Delta z = P_z \Delta\tau$, which forces $P_z = 0$ on any closed candidate of positive duration. This is (b). Clause (a) is Proposition .

The identities follow from the definitions of E and J in Theorem (a): $\dot{t} = E - \alpha e^{\beta x}\dot{\varphi}$ and $\dot{\varphi} = Je^{-2\beta x} + \alpha E e^{-\beta x}$ rearrange to $\dot{t} = E - \alpha w$. The future orientation of Theorem is aligned with $u_{\mathrm{d}} = \partial_t$, and $E = \theta^0(\gamma)$, so a future-directed unit-timelike geodesic has $E > 0$.

Differentiating in proper time gives $\ddot{x} = -w\,\partial_x w$, so a constantly vanishing $\dot{x}$ forces either $K = 0$ and $w = 0$, or a non-critical turning point, which cannot remain at rest. The condition $w = 0$ is $Je^{-\beta x_0} = -\alpha E$. Then $\dot{\varphi} = 0$ and $\dot{t} = E$. The normalisation with $\dot{x} = P_z = 0$ forces $K = 0$, hence $E = \pm 1$ when $P_z = 0$. This is (c): the worldline is a coordinate-time translate at fixed (x, φ, z), and cannot close in t.

Now take $E > 0$, $P_z = 0$, and $K = E^2 - 1 > 0$. The potential $V(x) := w(x)^2$ has an interior zero if and only if J and αE have opposite signs, hence $J < 0$ since $\alpha > 0$ and $E > 0$. If $J \geq 0$, then $V(x) \geq \alpha^2 E^2$. For $\alpha > 1$ one has $K = E^2 - 1 < \alpha^2 E^2$, so cannot be satisfied. Thus every such geodesic has a potential well, and $K < \alpha^2 E^2$ holds automatically, so $x(\tau)$ remains in a compact interval and is periodic.

To compute the period and the increments, set $w = Je^{-\beta x} + \alpha E$. Then $|w| \leq \sqrt{K} < \alpha E$, so $w - \alpha E < 0$ and $\alpha E - w > 0$. Differentiating w along the geodesic gives $\mathrm{d}w/\mathrm{d}\tau = -\beta(w - \alpha E)\dot{x}$. On the half-orbit with $\dot{x} > 0$, the sign of β arranges that w increases from $-\sqrt{K}$ to $\sqrt{K}$ when $\beta > 0$, and decreases through the same interval when $\beta < 0$; in either case

$$\mathrm{d}\tau = \frac{\mathrm{d}w}{|\beta|(\alpha E - w)\sqrt{K - w^2}}$$

on that half-orbit, and the return half-orbit contributes the same integrals. Thus

$$T_x = \frac{2}{|\beta|}\int_{-\sqrt{K}}^{\sqrt{K}} \frac{\mathrm{d}w}{(\alpha E - w)\sqrt{K - w^2}}.$$

The substitution $w = \sqrt{K}\sin\theta$ converts the integrand to $(\alpha E - \sqrt{K}\sin\theta)^{-1}\,\mathrm{d}\theta$ on $[-\pi/2, \pi/2]$. That definite integral

equals $\pi/\sqrt{\alpha^2E^2 - K}$, because the integrand depends on $\sin\theta$ only through a shift that covers a half-period of the standard integral $\int_0^{2\pi} (\alpha E - \sqrt{K}\cos\psi)^{-1}\,\mathrm{d}\psi = 2\pi/\sqrt{\alpha^2E^2 - K}$. Since $\alpha^2E^2 - K = (\alpha^2 - 1)E^2 + 1$, this is . (The informal period $2\pi/(\beta\sqrt{K})$ is therefore incorrect, and is not used.)

The same measure yields

$$\Delta t = \frac{2}{|\beta|}\int_{-\sqrt{K}}^{\sqrt{K}} \frac{E - \alpha w}{(\alpha E - w)\sqrt{K - w^2}}\,\mathrm{d}w.$$

Polynomial division gives $(E - \alpha w)/(\alpha E - w) = \alpha + (1 - \alpha^2)E/(\alpha E - w)$, and $\int_{-\sqrt{K}}^{\sqrt{K}} (K - w^2)^{-1/2}\,\mathrm{d}w = \pi$, so

$$\Delta t = \frac{2\pi}{|\beta|}\left(\alpha + \frac{(1 - \alpha^2)E}{\sqrt{(\alpha^2 - 1)E^2 + 1}}\right),$$

which is . The second summand is negative for $\alpha > 1$, but its absolute value is strictly smaller than α:

$$\frac{(\alpha^2 - 1)E}{\sqrt{(\alpha^2 - 1)E^2 + 1}} < \frac{(\alpha^2 - 1)E}{\sqrt{(\alpha^2 - 1)E^2}} = \sqrt{\alpha^2 - 1} < \alpha.$$

Hence $\Delta t > 0$ strictly. (The past-directed class $E < 0$ is the time-reverse of a future-directed geodesic, and a closed curve remains closed under reversal; it is enough to treat $E > 0$.)

Finally, $\dot{\varphi} = e^{-\beta x}w = w(w - \alpha E)/J$ when $J \neq 0$. Inserting the same measure, the factor $w - \alpha E$ cancels against the denominator (up to a sign absorbed in $\alpha E - w$), and $\Delta\varphi$ reduces to a constant multiple of the odd integral $\int_{-\sqrt{K}}^{\sqrt{K}} w(K - w^2)^{-1/2}\,\mathrm{d}w = 0$. Thus $\Delta\varphi = 0$ over each bounce, and after any integer number of bounces one still has $\Delta t \neq 0$. This is (d). Clause (e) is immediate from $\Delta z = P_z\Delta\tau$ together with the comparison $K - \alpha^2E^2 = -((\alpha^2 - 1)E^2 + 1 - P_z^2)$: a runaway would require $P_z^2 \geq (\alpha^2 - 1)E^2 + 1 > 0$.

Geodesic completeness of $(M, g_{\alpha,\beta})$ is already Theorem (a), so there are no incomplete timelike orbits to typestate as an additional closed class.

It remains to classify null geodesics, working throughout with an affine parameter and with the official first integral of Theorem (a) at $\varepsilon = 0$, which is . The identities are unchanged. A future-directed null geodesic has $E = \theta^0(\dot{\gamma}) > 0$.

If $E = 0$, then $K_0 = -P_z^2 \le 0$. The left-hand side of is nonnegative, so $P_z = 0$, $\dot{x} = 0$, and $w = 0$, hence $J = 0$. Then $\dot{t} = \dot{\varphi} = \dot{z} = 0$, which is the zero tangent. This is (a′).

Constant-$x \equiv 0$ again forces either $K_0 = 0$ and $w = 0$, or a non-critical turning point, which cannot remain at rest. The condition $w = 0$ is $Je^{-\beta x_0} = -\alpha E$, whence $\dot{\varphi} = 0$ and $\dot{t} = E$, while $K_0 = 0$ forces $P_z = \pm E$. If $E \neq 0$, both $\Delta t = E\Delta\lambda$ and $\Delta z = P_z\Delta\lambda$ are nonzero on any positive affine interval. If $K_0 > 0$ and $\dot{x} \equiv 0$, then $w = \pm\sqrt{K_0}$ would have to be a critical point of w, hence $w = \alpha E$ and $J = 0$; but then $K_0 = \alpha^2E^2 = E^2 - P_z^2$, so $\dot{x}^2 = -(P_z^2 + (\alpha^2 - 1)E^2) < 0$ for $\alpha > 1$, which is impossible. This is (b′).

Now take $E > 0$ and $P_z = 0$, so $K_0 = E^2 > 0$. As in the timelike argument, a potential well exists if and only if $J < 0$. If $J \ge 0$, then $V(x) \ge \alpha^2E^2 > E^2 = K_0$ for $\alpha > 1$, and cannot be satisfied. Thus x remains in a compact interval and is periodic. The same contour integral in w, now over $[-\sqrt{K_0}, \sqrt{K_0}] = [-E, E]$, applies, because $|w| \le E < \alpha E$ still guarantees $\alpha E - w > 0$. The official period integral therefore yields

$$T_{x,\mathrm{null}} = \frac{2\pi}{|\beta|\sqrt{\alpha^2E^2 - K_0}} = \frac{2\pi}{|\beta|E\sqrt{\alpha^2 - 1}},$$

which is . The same division $(E - \alpha w)/(\alpha E - w) = \alpha + (1 - \alpha^2)E/(\alpha E - w)$ gives

$$\Delta t_{\mathrm{null}} = \frac{2\pi}{|\beta|}\left(\alpha + \frac{(1 - \alpha^2)E}{E\sqrt{\alpha^2 - 1}}\right) = \frac{2\pi}{|\beta|}(\alpha - \sqrt{\alpha^2 - 1}),$$

which is . The identity $\alpha - \sqrt{\alpha^2 - 1} > 0$ holds for every $\alpha > 1$, and the increment is independent of the affine scale E. The net azimuthal increment again reduces to a constant multiple of the odd integral $\int_{-E}^{E} w(E^2 - w^2)^{-1/2}\,\mathrm{d}w = 0$. The past-directed class $E < 0$ is the time-reverse of a future-directed null geodesic, and a closed curve remains closed under reversal; it is enough to treat $E > 0$. This is (c′).

Finally, if $P_z \neq 0$, then $\Delta z = P_z\Delta\lambda \neq 0$ on any positive affine interval. Moreover $K_0 = E^2 - P_z^2 \le E^2 < \alpha^2E^2$ for $\alpha > 1$, so there is no runaway null radial motion (a well exists whenever $J < 0$ and $K_0 > 0$, while $J \ge 0$ is incompatible with the radial identity). This is (d′). The classification does not enlarge the geodesic-completeness typestate

already recorded in Theorem (a).

Corollary (Holonomy-shifted circular geodesics in a restricted class). *For a real constant A_0, define on M the holonomy-shifted metric*

$$g_{\alpha,\beta,A_0} := -(\mathrm{d}t + (A_0 + \alpha e^{\beta x})\mathrm{d}\varphi)^2 + \mathrm{d}x^2 + e^{2\beta x}\mathrm{d}\varphi^2 + \mathrm{d}z^2.$$

This metric is locally isometric to $g_{\alpha,\beta}$ (the shift $\hat{t} = t + A_0\varphi$ is single-valued only on simply connected opens, not on the quotient), is geodesically complete, and has the same constant orthonormal Einstein tensor . If $0 < \alpha < 1$ and $A_0 > 0$, then at the unique radius $x_* = \beta^{-1}\ln(\alpha A_0/(1-\alpha^2))$ one has

$$g_{\alpha,\beta,A_0}(\partial_\varphi, \partial_\varphi)|_{x_*} = -\frac{A_0^2}{1-\alpha^2} < 0$$

and $\partial_x g_{\alpha,\beta,A_0}(\partial_\varphi, \partial_\varphi)|_{x_*} = 0$. Hence every azimuthal circle at that radius is a closed future-directed timelike geodesic, of proper period $2\pi A_0/\sqrt{1-\alpha^2}$. This geodesic class is not a positive-dust Einstein match in the longitudinal-scalar branch of Theorem , because that branch requires $\alpha > 1$ for $\rho_{\mathrm{d}} > 0$.

Proof. Locally, $\hat{t} = t + A_0\varphi$ pulls back to , so curvature, completeness on the cover, and are preserved. Writing $y = e^{\beta x} > 0$ and $C(x) := g_{\alpha,\beta,A_0}(\partial_\varphi, \partial_\varphi) = (1-\alpha^2)y^2 - 2\alpha A_0 y - A_0^2$, the unique positive critical point in the stated class is $y_* = \alpha A_0/(1-\alpha^2)$, at which $C(x_*) = -A_0^2/(1-\alpha^2) < 0$ and $\nabla_{\partial_\varphi}\partial_\varphi = -\frac{1}{2}\mathrm{grad}\, C$ vanishes. Future orientation follows from $(A_0 + \alpha y_*) = A_0/(1-\alpha^2) > 0$.

Named Einstein-match branches

An identity of the form $M_{\mathrm{P}}^2(G_{ab}[g_{\alpha,\beta}] + \Lambda g_{\alpha,\beta,ab}) = T_{ab}$ is host-compatibility bookkeeping on a prescribed ansatz. It is not Einstein dynamics, not a derived metric, and not Host C. Each admitted branch is typestated by the sources that actually appear in the equation. A later rotating-family match does not cancel the flat residual of official Chrono-DM-A, and a dust-plus-negative-Λ Gödel-type match does not withdraw the official azimuthal winding residual [1]. Negative Λ is typestated as a negative cosmological term, not as "strictly positive-source" language.

Throughout this section the Einstein equation is written in the orthonormal frame of Theorem . Dust is always $T_{\mathrm{d}}{}_{ab} = \rho_{\mathrm{d}} u_a u_b$ with $u = e_0$, so that the only nonzero component is $T_{\mathrm{d}}{}_{\hat{0}\hat{0}} = \rho_{\mathrm{d}}$.

Dust plus a negative cosmological term at the Gödel value

Theorem (Gödel-type dust-plus-negative-Λ match). *Set $\alpha = \sqrt{2}$ and $\beta \neq 0$. With*

$$\Lambda = -\frac{1}{2}\beta^2 < 0, \qquad \rho_{\mathrm{d}} = M_{\mathrm{P}}^2 \beta^2 > 0,$$

the identity

$$M_{\mathrm{P}}^2 (G_{ab}[g_{\sqrt{2},\beta}] + \Lambda g_{\sqrt{2},\beta,ab}) = \rho_{\mathrm{d}} u_a u_b$$

holds everywhere on M. Every azimuthal circle is a closed future-directed host-timelike curve, by Proposition , and is not a geodesic. This is the official Chrono-DM-A typestate of the standard Gödel comparison, written here for the printed two-parameter family: it is not a new withdrawal of the official winding residual, and it is not a strictly positive-source match.

Proof. Theorem at $\alpha = \sqrt{2}$ gives $G_{\hat{a}\hat{b}} = \frac{1}{2}\beta^2 \mathrm{diag}\,(+1, +1, +1, +1)$. Adding $\Lambda \eta_{\hat{a}\hat{b}}$ with $\Lambda = -\frac{1}{2}\beta^2$ produces $G_{\hat{0}\hat{0}} - \Lambda = \beta^2$ and $G_{\widehat{\mathrm{imath}}\,\widehat{\mathrm{imath}}} + \Lambda = 0$ for imath $= 1, 2, 3$. Multiplying by M_{P}^2 yields . The causal typestate of the circles is Proposition .

Remark (Normalisation dictionary with official Chrono-DM-A). Official Chrono-DM-A records the unit-radius Gödel metric

$$g_{\mathrm{G}} = -\mathrm{d}t^2 - 2e^x \,\mathrm{d}t\,\mathrm{d}\psi - \frac{1}{2}e^{2x}\,\mathrm{d}\psi^2 + \mathrm{d}x^2 + \mathrm{d}z^2$$

on the cover, with the dust-plus-negative-Λ identity $G_{ab} + \Lambda g_{\mathrm{G},ab} = u_a u_b$ at $\Lambda = -\frac{1}{2}$ [1]. The change of angular coordinate $\varphi = \psi/\sqrt{2}$ identifies g_{G} with $g_{\sqrt{2},1}$ as printed in . For a general $\beta \neq 0$, the constant rescaling $(t, x, \varphi, z) \mapsto (\beta t, \beta x, \varphi, \beta z)$ takes $g_{\sqrt{2},\beta}$ to a constant multiple of g_{G}, and converts $\Lambda = -\frac{1}{2}\beta^2$, $\rho_{\mathrm{d}} = M_{\mathrm{P}}^2\beta^2$ into the official unit-radius normalisation. Theorem is therefore the same comparison, written for the two-parameter family, and

not a new Einstein identity.

Proposition (Canonical azimuthal winding residual on the rotating family). *Let $q = \kappa\,\mathrm{d}\varphi$ with $\kappa \neq 0$, and let $T_{\Theta}_ab = M_{\Theta}^2(q_a q_b - \frac{1}{2}g_{\alpha,\beta,ab}\sigma)$ be the canonical winding stress of , with $\sigma = \kappa^2 e^{-2\beta x}$.*

1. *For constant $M_{\Theta} > 0$, the orthonormal components*

$$T_{\Theta,\hat{0}\hat{0}} = -T_{\Theta,\hat{1}\hat{1}} = T_{\Theta,\hat{2}\hat{2}} = -T_{\Theta,\hat{3}\hat{3}} = \frac{1}{2}M_{\Theta}^2\sigma$$

are x-dependent, and $T_{\Theta,\hat{1}\hat{1}} = -T_{\Theta,\hat{2}\hat{2}}$. No choice of constants Λ and ρ_{d} satisfies

$$M_{\mathrm{P}}^2(G_{ab} + \Lambda g_{\alpha,\beta,ab}) = \rho_{\mathrm{d}} u_a u_b + T_{\Theta}_ab$$

on the whole of M.

2. *If the stiffness is modulated as $M_{\Theta}(x) = M_{\Theta,0}e^{\beta x}$, then $M_{\Theta}(x)^2\sigma = \mu^2 := M_{\Theta,0}^2\kappa^2$ is constant, and becomes constant, but the anisotropy $T_{\Theta,\hat{1}\hat{1}} = -T_{\Theta,\hat{2}\hat{2}}$ remains. Since $G_{\hat{1}\hat{1}} = G_{\hat{2}\hat{2}}$, the same obstruction persists: a residual-free match including this azimuthal T_{Θ} is not obtained.*

3. *In the test-stiffness limit $M_{\Theta} \to 0$ with $B\kappa^2$ held finite, one has $T_{\Theta} \to 0$ while the disformal deformation $Bq \otimes q$ may remain macroscopic. That limit drops the winding stress from the Einstein equation; it is not a winding-inclusive residual-free match. At the Gödel value of Theorem , this is the same obstruction already recorded as official Chrono-DM-A Proposition prop:Godelres [1]. That official residual is not withdrawn.*

Proof. From , $q^a = \kappa g^{a\varphi}$ has $\hat{q}^{\hat{2}} = \kappa e^{-\beta x}$ as its only orthonormal component, so $\sigma = \kappa^2 e^{-2\beta x}$ and follows from . For constant M_{Θ}, the (x,x) component of any linear combination of G_{ab}, Λg_{ab}, and $u_a u_b$ is constant, whereas T_{Θ}_xx is nonconstant. For the modulated stiffness, $\nabla_a(M_{\Theta}(x)^2 q^a) = 0$ holds identically (the only potentially nonzero contribution is $\frac{1}{\sqrt{-g}}\partial_x(\sqrt{-g}\,M_{\Theta}(x)^2 q^x) = 0$), and μ^2 is constant, but $T_{\Theta,\hat{1}\hat{1}} - T_{\Theta,\hat{2}\hat{2}} = -\mu^2 \neq 0$, while $G_{\hat{1}\hat{1}} - G_{\hat{2}\hat{2}} = 0$. No cosmological term, which is proportional to $\eta_{\hat{a}\hat{b}}$, and no dust term, which affects only the $(\hat{0}\hat{0})$ slot, can cancel that difference.

Proposition (A second closed one-form on printed $g_{\alpha,\beta}$). *Keep the official metric $g_{\alpha,\beta}$ unchanged, and keep the modulated azimuthal controller $q_{(\varphi)} = \kappa\,\mathrm{d}\varphi$ of Proposition (ii), with constant $\mu_{\varphi}^{2} := M_{\Theta,0}^{2}\kappa^{2} > 0$ and orthonormal stress $\frac{1}{2}\mu_{\varphi}^{2}\mathrm{diag}\,(+1,-1,+1,-1)$. Adjoin a second closed one-form $q_{(x)} := \kappa_{x}\,\mathrm{d}x = \kappa_{x}\theta^{1}$ with κ_{x} constant and with the canonical stress .*

1. *If the stiffness M_{x} is constant, then $\sigma_{x} = \kappa_{x}^{2}$ is constant and the orthonormal stress is $\frac{1}{2}\mu_{x}^{2}\mathrm{diag}\,(+1,+1,-1,-1)$ with $\mu_{x}^{2} := M_{x}^{2}\kappa_{x}^{2}$. At $\mu_{x}^{2} = \mu_{\varphi}^{2} =: \mu^{2} > 0$ the formal sum is*

$$T_{q_{(\varphi)},\,\hat{a}\hat{b}} + T_{q_{(x)},\,\hat{a}\hat{b}} = \mu^{2}\mathrm{diag}\,(+1,0,0,-1).$$

This diagonal is not the official Maxwell stress of Lemma , which is $\frac{1}{2}\mathscr{B}_{0}^{2}\mathrm{diag}\,(+1,+1,+1,-1)$. The current identity $\nabla_{a}(M_{x}^{2}q_{(x)}^{a}) = e^{-\beta x}\partial_{x}(e^{\beta x}M_{x}^{2}\kappa_{x}) = \beta M_{x}^{2}\kappa_{x}$ fails to vanish, because official $\sqrt{-g} = e^{\beta x}$.

2. *If the stiffness is modulated as $M_{x}(x) = M_{x,0}e^{-\beta x/2}$, then $\nabla_{a}(M_{x}(x)^{2}q_{(x)}^{a}) = 0$ holds identically, but $\mu_{x}^{2}(x) = M_{x,0}^{2}\kappa_{x}^{2}e^{-\beta x}$ is nonconstant. The resulting x-dependent stress cannot cancel the constant anisotropy of the azimuthal slot, nor match the constant Einstein tensor of Theorem , on the whole of M.*

3. *Consequently, current conservation and a constant two-form stress cannot be satisfied simultaneously for $q_{(x)}$ with constant κ_{x}. A complete on-shell residual-free Einstein match, in which both controller currents are conserved and the printed Einstein equation holds everywhere on M, is not obtained. The algebraic assignment $\Lambda = -\alpha^{2}\beta^{2}/4$, $\mu^{2} = \frac{1}{2}M_{\mathrm{P}}^{2}\beta^{2}(\alpha^{2}-2)$, and $\rho_{\mathrm{d}} = \frac{1}{2}M_{\mathrm{P}}^{2}\beta^{2}\alpha^{2}$, which would formally match $M_{\mathrm{P}}^{2}(G_{ab} + \Lambda g_{ab})$ to geodesic dust plus for $\alpha > \sqrt{2}$, is therefore not admitted as an official branch: the x-current is off-shell at constant stiffness, the cosmological term is not the official value $\Lambda = -\frac{1}{2}\beta^{2}$ of Branches G, L, and M except at the degenerate point $\alpha = \sqrt{2}$ where $\mu^{2} = 0$, and official Branch M parameters are not to be pasted. The single-slot residual of Proposition is not withdrawn. Official Branches G, L, and M are not withdrawn. Official Branch L remains dust plus a longitudinal $q_{(z)} = \kappa_{z}\theta^{3}$ for $\alpha \in (1,\sqrt{2}]$, not an azimuthal-plus-$\mathrm{d}x$ pair. $\Lambda = 0$ is not obtained on $g_{\alpha,\beta}$. The inventory is not Host C and is not Bar 3.*

Proof. The azimuthal stress is Proposition (ii). For $q_{(x)} = \kappa_x \, \mathrm{d}x$ one has $q^{\hat{1}} = \kappa_x$ as the only orthonormal component and $\sigma_x = \kappa_x^2$, so produces $\frac{1}{2}\mu_x^2 \mathrm{diag}\,(+1, +1, -1, -1)$ at constant M_x. Adding the azimuthal diagonal gives . The official Maxwell stress is , whose spatial signature on the $(\hat{1}, \hat{2})$ plane is isotropic and positive, whereas vanishes in those two slots and is negative in the $\hat{3}$ slot. The divergence identity in (i) is the official volume element $\sqrt{-g} = e^{\beta x}$ together with $q^x_{(x)} = \kappa_x$ and $q^t_{(x)} = q^{\varphi}_{(x)} = q^z_{(x)} = 0$. For the profile in (ii), $e^{\beta x} M_x(x)^2 \kappa_x = M^2_{x,0} \kappa_x$ is constant, so the same divergence vanishes, while μ_x^2 retains the factor $e^{-\beta x}$.

If, ignoring the off-shell current, one formally equates $M_{\mathrm{P}}^2 (G_{\hat{a}\hat{b}} + \Lambda \eta_{\hat{a}\hat{b}})$ to ρ_{d} in the $(\hat{0}\hat{0})$ slot plus , the $(\hat{1}\hat{1})$ and $(\hat{2}\hat{2})$ equations force $\Lambda = -\alpha^2\beta^2/4$, the $(\hat{3}\hat{3})$ equation then forces $\mu^2 = \frac{1}{2} M_{\mathrm{P}}^2 \beta^2 (\alpha^2 - 2)$, and the $(\hat{0}\hat{0})$ equation forces $\rho_{\mathrm{d}} = \frac{1}{2} M_{\mathrm{P}}^2 \beta^2 \alpha^2$. These are not official Branch M parameters, and they are not admitted as a residual-free on-shell identity, by the current obstruction in (i)–(ii).

Longitudinal closed one-form

Theorem (Dust plus longitudinal scalar plus negative Λ). *Fix $\alpha \in (1, \sqrt{2}\,]$ and $\beta \neq 0$. Let $q_{(z)} = \kappa_z \, \mathrm{d}z = \kappa_z \theta^3$ with constant κ_z, and let $T_{q_{(z)}}$ be the canonical stress of this closed one-form (with $M_\Theta = 1$). The assignment*

$$\rho_{\mathrm{d}} = M_{\mathrm{P}}^2 \beta^2 (\alpha^2 - 1) > 0, \qquad \kappa_z^2 = \tfrac{1}{2} M_{\mathrm{P}}^2 \beta^2 (2 - \alpha^2) \geq 0, \qquad \Lambda = -\tfrac{1}{2}\beta^2 < 0$$

solves

$$M_{\mathrm{P}}^2 (G_{ab}[g_{\alpha,\beta}] + \Lambda g_{\alpha,\beta,ab}) = \rho_{\mathrm{d}} u_a u_b + T_{q_{(z)},ab}$$

identically on all of M. The one-form is closed and co-closed: $\mathrm{d}q_{(z)} = 0$ *and* $\nabla_a q^a_{(z)} = 0$. *At $\alpha = \sqrt{2}$ one has $\kappa_z = 0$, and the branch reduces to Theorem . This is a different source class from canonical azimuthal winding: $q_{(z)}$ is not the disformal controller used to shield the ordinary sector in Section . The cosmological term remains strictly negative.*

Proof. The Hodge dual on the oriented coframe satisfies $\star_g \theta^3 = \theta^0 \wedge \theta^1 \wedge \theta^2 = e^{\beta x} \, \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}\varphi$, whose exterior

derivative vanishes identically, so $q_{(z)}$ is co-closed; it is closed because κ_z is constant. The stress is

$$T_{q_{(z)},\hat{a}\hat{b}} = \tfrac{1}{2}\kappa_z^2 \mathrm{diag}\,(+1,-1,-1,+1).$$

The orthonormal Einstein equation therefore reduces to the three independent scalar conditions

$$M_{\mathrm{P}}^2\left(\frac{\beta^2\alpha^2}{4} + \Lambda\right) = -\tfrac{1}{2}\kappa_z^2,$$

$$M_{\mathrm{P}}^2\left(\frac{\beta^2}{4}(4-\alpha^2) + \Lambda\right) = +\tfrac{1}{2}\kappa_z^2,$$

$$M_{\mathrm{P}}^2\left(\frac{\beta^2}{4}(3\alpha^2-4) - \Lambda\right) = \rho_{\mathrm{d}} + \tfrac{1}{2}\kappa_z^2.$$

Subtracting from gives $\kappa_z^2 = \tfrac{1}{2}M_{\mathrm{P}}^2\beta^2(2-\alpha^2)$, which is non-negative precisely for $\alpha^2 \le 2$. Adding those two equations gives $2\Lambda + \beta^2 = 0$, hence $\Lambda = -\tfrac{1}{2}\beta^2$. Substitution into produces

$$\rho_{\mathrm{d}} = M_{\mathrm{P}}^2\beta^2\left(\frac{3\alpha^2-4}{4} + \frac{1}{2} - \frac{2-\alpha^2}{4}\right) = M_{\mathrm{P}}^2\beta^2(\alpha^2-1).$$

For $\alpha > 1$ the density is strictly positive, and at $\alpha = \sqrt{2}$ one recovers .

Force-free charged dust plus Maxwell

The Version 1 Maxwell two-form $\mathscr{F} = \mathscr{B}_0\theta^1 \wedge \theta^2$ is closed, but it is not co-closed, and the Version 1 parameter set does not solve the Einstein equation except at the Gödel point where the Maxwell field vanishes. The correct match is recorded here as a charged-dust source class.

Lemma (Maxwell field, current, and stress). *Let* $\mathscr{F} = \mathscr{B}_0\,\theta^1 \wedge \theta^2 = \mathscr{B}_0 e^{\beta x}\,\mathrm{d}x \wedge \mathrm{d}\varphi$ *with constant* $\mathscr{B}_0$. *Then* $\mathrm{d}\mathscr{F} = 0$, *while*

$$\nabla_a \mathscr{F}^{ab} = -\alpha\beta\mathscr{B}_0\, e_0^{\,b}.$$

The Hilbert stress $T_{\mathscr{F},ab} = \mathscr{F}_{ac}\mathscr{F}_{b}{}^{c} - \tfrac{1}{4}g_{ab}\mathscr{F}_{cd}\mathscr{F}^{cd}$ *has orthonormal components*

$$T_{\mathscr{F},\hat{a}\hat{b}} = \tfrac{1}{2}\mathscr{B}_0^2\,\mathrm{diag}\,(+1,+1,+1,-1).$$

The Lorentz force on $u = e_0$ vanishes identically: $\iota_{e_0}\mathscr{F} = 0$. Thus e_0 remains geodesic, as already recorded in Theorem (c).

Proof. Closure is immediate from $\mathrm{d}(e^{\beta x}\mathrm{d}x \wedge \mathrm{d}\varphi) = 0$. The only nonzero coordinate components are $F_{x\varphi} = \mathscr{B}_0 e^{\beta x}$, and the inverse metric gives $F^{xt} = -\alpha\mathscr{B}_0$ and $F^{x\varphi} = \mathscr{B}_0 e^{-\beta x}$. With $\sqrt{-g} = e^{\beta x}$,

$$\nabla_a \mathscr{F}^{at} = \frac{1}{\sqrt{-g}}\partial_x(\sqrt{-g}\,F^{xt}) = -\alpha\beta\mathscr{B}_0,$$

and the remaining components of $\nabla_a \mathscr{F}^{ab}$ vanish, which is . The stress evaluation in the orthonormal frame is the standard Maxwell computation for a single component $F_{\hat{1}\hat{2}} = \mathscr{B}_0$, using $\mathscr{F}_{cd}\mathscr{F}^{cd} = 2\mathscr{B}_0^2$. Finally, $\mathscr{F}$ has no legs along θ^0, so $\iota_{e_0}\mathscr{F} = 0$.

Theorem (Force-free charged dust plus Maxwell plus negative Λ). *Fix $\alpha \in [\sqrt{2}, 2]$ and $\beta \neq 0$, and let $\mathscr{F}$ be as in Lemma . The assignment*

$$\rho_{\mathrm{d}} = \tfrac{1}{2}M_{\mathrm{P}}^2\beta^2\alpha^2 > 0, \qquad \mathscr{B}_0^2 = \tfrac{1}{2}M_{\mathrm{P}}^2\beta^2(\alpha^2 - 2) \geq 0, \qquad \Lambda = -\tfrac{1}{2}\beta^2 < 0$$

solves

$$M_{\mathrm{P}}^2(G_{ab}[g_{\alpha,\beta}] + \Lambda g_{\alpha,\beta,ab}) = \rho_{\mathrm{d}} u_a u_b + T_{\mathscr{F},ab}$$

identically on all of M. The Maxwell current is parallel to the dust velocity, so the dark fluid carries a force-free dark $U(1)_D$ charge density $-\alpha\beta\mathscr{B}_0$. This is not a source-free Maxwell field, and it is not a winding-inclusive match. At $\alpha = \sqrt{2}$ one has $\mathscr{B}_0 = 0$, and the branch reduces to Theorem . At $\alpha = 2$ one still has $\Lambda = -\frac{1}{2}\beta^2 \neq 0$, together with $\rho_{\mathrm{d}} = 2M_{\mathrm{P}}^2\beta^2$ and $\mathscr{B}_0^2 = M_{\mathrm{P}}^2\beta^2$. In particular, $\Lambda = 0$ is not obtained in this class, and the Version 1 parameter set

$$\rho_{\mathrm{d}}\mathrm{stackrel\,v1} = M_{\mathrm{P}}^2\beta^2(2 - \tfrac{1}{2}\alpha^2), \qquad \mathscr{B}_0^2\mathrm{stackrel\,v1} = \tfrac{1}{2}M_{\mathrm{P}}^2\beta^2(\alpha^2 - 2), \qquad \Lambda\mathrm{stackrel\,v1} = -\tfrac{\beta^2}{4}(4 - \alpha^2)$$

does not satisfy except at $\alpha = \sqrt{2}$, where the Maxwell field vanishes. The Som–Raychaudhuri charged-dust spacetime is a named comparison, not a licence to style $\alpha = 2$ as a $\Lambda = 0$ residual-free match on the printed homogeneous family.

Proof. With , the orthonormal Einstein equation is

$$M_{\mathrm{P}}^2\left(\frac{\alpha^2\beta^2}{4}+\Lambda\right)=\frac{1}{2}\mathscr{B}_0^2,$$

$$M_{\mathrm{P}}^2\left(\frac{\beta^2}{4}(4-\alpha^2)+\Lambda\right)=-\frac{1}{2}\mathscr{B}_0^2,$$

$$M_{\mathrm{P}}^2\left(\frac{\beta^2}{4}(3\alpha^2-4)-\Lambda\right)=\rho_{\mathrm{d}}+\frac{1}{2}\mathscr{B}_0^2.$$

Adding and forces $\Lambda=-\frac{1}{2}\beta^2$, independently of α. Back-substitution into gives $\mathscr{B}_0^2=\frac{1}{2}M_{\mathrm{P}}^2\beta^2(\alpha^2-2)$, which is non-negative precisely for $\alpha^2\geq 2$. Then produces $\rho_{\mathrm{d}}=\frac{1}{2}M_{\mathrm{P}}^2\beta^2\alpha^2$. The Version 1 set instead uses $\Lambda_{\mathrm{v1}}=-\frac{\beta^2}{4}(4-\alpha^2)$. With that choice,

$$M_{\mathrm{P}}^2(G_{\widehat{33}}+\Lambda_{\mathrm{v1}})=0,\ -\frac{1}{2}\mathscr{B}_{0,\mathrm{v1}}^2=-\frac{1}{4}M_{\mathrm{P}}^2\beta^2(\alpha^2-2),$$

which agree only at $\alpha=\sqrt{2}$. The same mismatch appears in the $(\widehat{00})$ and $(\widehat{11})$ slots, and is recorded in Appendix . Force-freeness and the current are Lemma .

Remark (What is and is not withdrawn). The three admitted branches are residual-free for the sources actually written in , , and . They do not withdraw official Chrono-DM-A Proposition prop:flatres, nor Proposition prop:Godelres [1]. They do not produce a four-slot Einstein identity in which the disformal azimuthal controller stress T_{Θ} of Proposition is present and cancelled. Proposition does not supply such an identity either, and does not withdraw the three admitted branches. They do not convert a negative cosmological term into a strictly positive source. A vanishing dust density is not claimed: every admitted branch with α in the stated range has $\rho_{\mathrm{d}}>0$.

Proposition (Energy typestate of the admitted source classes). *On every admitted branch, the named matter stresses on the right-hand side of the Einstein equation satisfy the host null energy condition, while the cosmological term is typestated separately as a strictly negative Λ.*

1. *Geodesic dust with $\rho_{\mathrm{d}}>0$ satisfies $T_{\mathrm{d}}{}_{ab}k^ak^b=\rho_{\mathrm{d}}(u\cdot k)^2\geq 0$ for every g-null k.*

2. *The longitudinal stress of Theorem satisfies $T_{q_{(z)},ab}k^ak^b=(q_{(z)}\cdot k)^2\geq 0$ for every g-null k, even though its spatial*

diagonal has tension $T_{q_{(z)},\hat{11}} = T_{q_{(z)},\hat{22}} = -\frac{1}{2}\kappa_z^2 \le 0$.

3. *The Maxwell stress of Theorem satisfies the standard null and weak energy conditions of source-free Maxwell theory on the stress tensor itself; the Maxwell field is nonetheless sourced, by Lemma .*

4. *If the cosmological term is moved to the right-hand side as* $T_{\Lambda,ab} := -M_{\mathrm{P}}^2 \Lambda g_{ab}$*, then* $\Lambda = -\frac{1}{2}\beta^2 < 0$ *gives a strictly negative energy density* $T_{\Lambda,\hat{00}} = -\frac{1}{2}M_{\mathrm{P}}^2\beta^2 < 0$*. This is why no admitted branch is styled as a strictly positive-source match.*

Proof. The dust and winding-stress contractions are the same evaluations as in Theorem (d). For the Maxwell stress and a g-null k, the contraction $T_{\mathscr{F},ab}k^a k^b = \frac{1}{2}\mathscr{B}_0^2((\hat{k^0})^2 + (\hat{k^1})^2 + (\hat{k^2})^2 - (\hat{k^3})^2)$ reduces, by $(\hat{k^0})^2 = (\hat{k^1})^2 + (\hat{k^2})^2 + (\hat{k^3})^2$, to $\mathscr{B}_0^2((\hat{k^1})^2 + (\hat{k^2})^2) \ge 0$. For a unit timelike v one has $\sum_{i=1}^{3}(\widehat{\mathrm{imath}})^2 = (\hat{v^0})^2 - 1$, and the same contraction expands to $\frac{1}{2}\mathscr{B}_0^2(1 + 2(\hat{v^1})^2 + 2(\hat{v^2})^2) \ge 0$. The cosmological accounting is the definition $T_{\Lambda,ab} := -M_{\mathrm{P}}^2\Lambda g_{ab}$ together with $\Lambda = -\frac{1}{2}\beta^2$.

Ordinary Cauchy foliation on the rotating host

The disformal ordinary metric on the rotating family uses the azimuthal controller $q = \kappa\,\mathrm{d}\varphi$, which is the chronology-split device, not the longitudinal one-form of Theorem . By Proposition , that controller's stress is either dropped in the test-stiffness limit or left as a residual at finite stiffness. The Cauchy typestate below is kinematic, on a prescribed weight $B(s(x))$.

Theorem (Ordinary Cauchy foliation, on the region where $H(x)$ grows). *Fix* $\alpha > 1$ *and* $\beta \neq 0$ *on* $M = \mathbb{R}_t \times \mathbb{R}_x \times S^1_\varphi \times \mathbb{R}_z$*, let* $q = \kappa\,\mathrm{d}\varphi$ *with* $\kappa > 0$*, and let* $B(s(x)) \ge 0$ *be a smooth disformal profile such that*

$$H(x) := B(s(x))\kappa^2 - (\alpha^2 - 1)e^{2\beta x} \ge \varepsilon_0\, e^{2\beta x}$$

holds for a constant $\varepsilon_0 > 0$ *and for all* x *in an open interval* $I \subseteq \mathbb{R}$*, possibly* $I = \mathbb{R}$*. On the open submanifold* $M_I := \mathbb{R}_t \times I \times S^1_\varphi \times \mathbb{R}_z$*, equipped with* $\hat{\tilde{g}} = g_{\alpha,\beta} + B(s(x))q \otimes q$*:*

1. *$\Delta_q(x) = 1 + B(s(x))\kappa^2 e^{-2\beta x} \geq \alpha^2 + \varepsilon_0 > 1$, so $\tilde{g}$ is smooth and Lorentzian on M_I;*

2. *completing the square gives*

$$\tilde{g} = -\left(1 + \frac{\alpha^2 e^{2\beta x}}{H(x)}\right)\mathrm{d}t^2 + H(x)\left(\mathrm{d}\varphi - \frac{\alpha e^{\beta x}}{H(x)}\,\mathrm{d}t\right)^2 + \mathrm{d}x^2 + \mathrm{d}z^2,$$

with uniformly bounded lapse $1 < N_g(x) := \sqrt{1 + \alpha^2 e^{2\beta x}/H(x)} \leq \sqrt{1 + \alpha^2/\varepsilon_0}$;

3. *if $I = \mathbb{R}$, then $(M, \tilde{g})$ is globally hyperbolic, and every level hypersurface $\Sigma_{t_0} := \{t = t_0\}$ is a smooth Cauchy surface for $(M, \tilde{g})$. If $I \neq \mathbb{R}$, the same Cauchy-slice argument applies to $(M_I, \tilde{g})$ only after the incomplete endpoints of I are typestated: a half-space or bounded slab is not a complete ordinary-Cauchy witness on all of M. Meanwhile, for $\alpha > 1$, the host $(M, g_{\alpha,\beta})$ has a closed future-directed timelike curve through every point, by Proposition . An exponential profile $B(s(x)) = B_0 e^{2\beta x}$ with $B_0\kappa^2 \geq \alpha^2 - 1 + \varepsilon_0$ realises on all of M. A constant profile $B \equiv B_0$ realises it only on a half-line in x (for $\beta > 0$, on $x \leq x_*$ for a finite x_*), which reintroduces the incomplete-collar obstruction already recorded in official Chrono-DM-A [1]. The profile is prescribed, not obtained from Einstein initial-value evolution.*

Proof. (i) Since $\sigma = g^{\varphi\varphi}\kappa^2 = \kappa^2 e^{-2\beta x}$, the bound gives $\Delta_q(x) = 1 + H(x)e^{-2\beta x} + (\alpha^2 - 1) \geq \alpha^2 + \varepsilon_0 > 1$. Lemma then gives Lorentzian signature.

(ii) Expanding the right-hand side of cancels the pair $\mp\frac{\alpha^2 e^{2\beta x}}{H(x)}\mathrm{d}t^2$ and leaves $-\mathrm{d}t^2 - 2\alpha e^{\beta x}\mathrm{d}t\,\mathrm{d}\varphi + H(x)\mathrm{d}\varphi^2 + \mathrm{d}x^2 + \mathrm{d}z^2$, which equals $g_{\alpha,\beta} + B(s(x))\kappa^2\mathrm{d}\varphi^2$. The lapse bound is immediate from $H(x)e^{-2\beta x} \geq \varepsilon_0$.

(iii) Along any non-constant future-directed $\tilde{g}$-causal curve $\gamma(\lambda) = (t(\lambda), x(\lambda), \varphi(\lambda), z(\lambda))$ in M_I, enforces

$$\dot{x}^2 + \dot{z}^2 + H(x)\left(\dot{\varphi} - \frac{\alpha e^{\beta x}}{H(x)}\dot{t}\right)^2 \leq N_g(x)^2\dot{t}^2 \leq N_{\max}^2\dot{t}^2,$$

with $N_{\max} := \sqrt{1 + \alpha^2/\varepsilon_0}$. In particular $\dot{t} > 0$, so t is a strict $\tilde{g}$-temporal function and γ may be reparameterised by t. If $I = \mathbb{R}$ and (t_-, t_+) is a bounded interval with $t_+ < \infty$, then $|x'(t)|^2 + |z'(t)|^2 \leq N_{\max}^2$, so $x(t)$ and $z(t)$ remain in a

compact rectangle and have finite arc length. On that rectangle, $H(x)$ is bounded above and bounded away from zero, so $|\varphi'(t)|$ is likewise bounded, and $(x(t), \varphi(t), z(t))$ converges as $t$uparrow t_+ in the complete Riemannian slice $\mathbb{R}_x \times S^1_\varphi \times \mathbb{R}_z$, contradicting future inextendibility. Symmetrically $t_- = -\infty$. Every inextendible $\tilde{g}$-causal curve therefore intersects every slice Σ_{t_0} exactly once. Compactness of causal diamonds $J^+_{\tilde{g}}(p) \cap J^-_{\tilde{g}}(q)$ follows from the same uniform spatial Lipschitz bounds on $[t(p), t(q)]$. If $I \neq \mathbb{R}$, the same estimates hold only up to the endpoints of I; those endpoints are not supplied as complete Cauchy boundaries in this sheet.

Lemma (On-shell test-switch profiles for the exponential ordinary shield). *Let $\alpha > 1$, $\beta \neq 0$, and $\kappa > 0$, and work on $(M, g_{\alpha,\beta})$ with $\mathscr{T}^{ab} = 0$. For a smooth function $s = s(x)$ of the Cartesian coordinate x only, the official volume element $\sqrt{-g} = e^{\beta x}$ yields*

$$\mathrm{Box}_g\, s = s'' + \beta s'.$$

The switch equation then reduces to $F_s^2 \mathrm{Box}_g\, s - W'(s) = 0$. The following two profiles are typestated separately; they are not a single "vacuum shield".

1. *(Linear switch, potential unbounded below.) The linear profile $s(x) = s_0 x$ with $s_0 \neq 0$, the exponential coupling $B(s) = B_0 \exp((2\beta/s_0)s)$ for a constant $B_0 \geq 0$, and the linear potential $W(s) = F_s^2 \beta s_0 s + W_0$ solve . The induced weight is $B(s(x)) = B_0 e^{2\beta x}$, and the growth bound holds on all of M whenever $B_0 \kappa^2 \geq \alpha^2 - 1 + \varepsilon_0$. The potential W is unbounded below on $\mathbb{R}$, and is not typestated as a non-negative potential.*

2. *(Constant non-negative potential.) The first-order profile $s'(x) = s_1 e^{-\beta x}$ with $s_1 \neq 0$ gives $\mathrm{Box}_g\, s \equiv 0$, and is solved by*

$$s(x) = s_\infty - \frac{s_1}{\beta} e^{-\beta x}$$

for a constant s_∞. The constant potential $W \equiv W_0 \geq 0$ then solves . On the image of , one has $s(x) \neq s_\infty$ at every finite x, and the coupling $B(s) = B_0 (s - s_\infty)^{-2}$ with $B_0 \geq 0$ is smooth and non-negative on all of M, with $B(s(x)) = B_0 (\beta/s_1)^2 e^{2\beta x}$. The growth bound holds on all of M whenever $B_0 (\beta/s_1)^2 \kappa^2 \geq \alpha^2 - 1 + \varepsilon_0$. As $s \to s_\infty$, which

occurs at the end of the x-line where $e^{-\beta x} \to 0$ *(namely* $x \to +\infty$ *if* $\beta > 0$*, and* $x \to -\infty$ *if* $\beta < 0$*), the weight* $B(s)$ *diverges. That divergence is at infinity in x, not at a finite-coordinate singularity; at the opposite end,* $B(s(x)) \to 0$*. Both profiles are on-shell test-switch realisations of the exponential weight already prescribed in Theorem . They are not Einstein back-reaction, not Host C, not a derived metric, and not Bar 3. The ordinary shield remains kinematic in the sense of that theorem.*

Proof. The wave operator on a scalar of x only is $\mathrm{Box}_g s = e^{-\beta x}\partial_x(e^{\beta x} g^{xx} s')$. Since $g^{xx} = 1$, this is . With $\mathscr{T}^{ab} = 0$, the last term of drops, leaving $F_s^2 \mathrm{Box}_g s - W'(s) = 0$.

For (i), $s'' = 0$ and $\mathrm{Box}_g s = \beta s_0$, so the switch equation forces $W'(s) = F_s^2 \beta s_0$, hence the printed linear potential. Then $B(s(x)) = B_0 \exp(2\beta x)$, and the comparison with is the same coefficient bound already recorded in Theorem . Because $\beta s_0 \neq 0$, the potential has a nonzero constant slope on all of $\mathbb{R}$, and is therefore unbounded below.

For (ii), $s'' = -\beta s_1 e^{-\beta x}$, so $\mathrm{Box}_g s = s'' + \beta s' = 0$. The switch equation then forces $W' = 0$, hence a constant potential, which may be taken non-negative. Integrating $s'(x) = s_1 e^{-\beta x}$ produces . The difference $s(x) - s_\infty = -(s_1/\beta)e^{-\beta x}$ never vanishes at finite x, so $B(s) = B_0(s - s_\infty)^{-2}$ is a smooth non-negative function of s along the entire trajectory, independently of the sign of $s - s_\infty$. Substituting the difference gives $B(s(x)) = B_0(\beta/s_1)^2 e^{2\beta x}$, and the growth bound follows as in (i). The end-of-the-line divergence and the decay at the opposite end are the two limits $e^{-\beta x} \to 0$ and $e^{-\beta x} \to +\infty$.

Proposition (Comoving ordinary sources in the switch equation). *Let* $\alpha > 1$*,* $\beta \neq 0$*, and* $\kappa > 0$*, and work on the official overlay* $(M, g_{\alpha,\beta}, \tilde{g})$ *with the azimuthal controller* $q = \kappa\, \mathrm{d}\varphi$ *of Theorem . Then* $q_a \partial_t^a = 0$ *and* $\tilde{g}(\partial_t, \partial_t) = -1$*, so the vector* $\tilde{u} := \partial_t$ *is unit timelike for both metrics. Write* $\mathscr{T}^{ab} = \sqrt{\Delta_q}\, \widehat{T}_{\mathrm{o}}^{ab}$ *as in Theorem .*

1. *For stationary comoving ordinary Jordan-frame dust* $\widehat{T}_{\mathrm{o}}^{ab} = \widehat{\rho}_{\mathrm{o}}(x)\, \tilde{u}^a \tilde{u}^b$*, one has* $\mathscr{T}^{ab} q_a q_b = 0$ *identically. Thus comoving dust, even when* $\mathscr{T}^{ab} \neq 0$*, drops out of the switch equation .*

2. *For a stationary comoving isotropic Jordan fluid* $\widehat{T}_{\mathrm{o}}^{ab} = (\widehat{\rho}_{\mathrm{o}} + \widehat{p}_{\mathrm{o}}) \tilde{u}^a \tilde{u}^b + \widehat{p}_{\mathrm{o}} \tilde{g}^{ab}$*, one has* $\widehat{T}_{\mathrm{o}}^{ab} q_a q_b = \widehat{p}_{\mathrm{o}} \sigma/\Delta_q$*, and*

therefore

$$\frac{1}{2}B'(s)\mathscr{T}^{ab}q_a q_b = \frac{B'(s)\sigma}{2\sqrt{\Delta_q}}\hat{p}_{\mathrm{o}} = \frac{B'(s)\kappa^2 e^{-2\beta x}}{2\sqrt{\Delta_q}}\hat{p}_{\mathrm{o}}.$$

3. *Because $\tilde{u} = \partial_t$ is $\tilde{g}$-geodesic (the metric is stationary and $\tilde{g}_{tt} = -1$ is constant) and has vanishing $\tilde{g}$-expansion, the Jordan Euler equation forbids a radially varying comoving pressure: $\tilde{g}^{xx} = 1$ would otherwise leave a nonzero projected pressure gradient. A radially decaying comoving $\hat{p}_{\mathrm{o}}(x)$ is therefore not Jordan–Euler-on-shell in this class.*

4. *If $\hat{p}_{\mathrm{o}}$ is a constant, the Jordan Euler and energy identities hold for any stationary density $\hat{\rho}_{\mathrm{o}}(x)$. Moreover $Q_\Theta = -\nabla_a(B(s)\mathscr{T}^{ab}q_b) = 0$ identically, because $\mathscr{T}^{ab}q_b = \hat{p}_{\mathrm{o}}q^a/\sqrt{\Delta_q}$ is purely azimuthal and temporal, and every coefficient is independent of (t, φ). On the official linear switch of Lemma (i), with exponential coupling and constant $\Delta_q = 1 + B_0\kappa^2$, the source is constant and is absorbed into a shifted linear potential, which remains unbounded below on $\mathbb{R}$. On the official first-order profile of Lemma (ii), a nonzero constant pressure replaces the constant potential by a logarithmic potential in $s - s_\infty$, which is not typestated as non-negative on the whole line. In both constant-pressure classes, the switch equation and the winding equation hold together with Jordan conservation, so the four-slot identity applies. The official vacuum profiles of Lemma remain, including that the linear potential is unbounded below and that the constant non-negative potential has an end-of-the-line divergence of $B(s)$. The kinematic Cauchy typestate of Theorem stands whenever the disformal weight still satisfies the official growth bound; a pressure source does not manufacture that weight. A pressure term is not Einstein back-reaction on $g_{\alpha,\beta}$, is not Host C, and is not Bar 3.*

Proof. The vanishing $q(\partial_t) = 0$ is immediate from $q = \kappa\,\mathrm{d}\varphi$, and $\tilde{g}_{tt} = g_{tt} = -1$ follows because $q_t = 0$. For comoving dust, $\hat{T}^{ab}_{\mathrm{o}}q_a q_b = \hat{\rho}_{\mathrm{o}}(q_a\tilde{u}^a)^2 = 0$, which is (i).

For an isotropic Jordan fluid, the same contraction reduces to $\hat{p}_{\mathrm{o}}\tilde{g}^{ab}q_a q_b$. Lemma gives $\tilde{g}^{ab}q_a q_b = \sigma - B\sigma^2/\Delta_q = \sigma/\Delta_q$, and multiplying by $\sqrt{\Delta_q}$ produces , which is (ii).

Stationarity and $\tilde{g}_{tt} = -1$ imply $\tilde{\Gamma}^{\lambda}_{tt} = -\frac{1}{2}\tilde{g}^{\lambda\sigma}\partial_\sigma \tilde{g}_{tt} = 0$, so $\tilde{\nabla}_u \tilde{u} = 0$. The expansion $\tilde{\nabla}_a \tilde{u}^a = \frac{1}{\sqrt{-\tilde{g}}}\partial_t(\sqrt{-\tilde{g}})$ vanishes for any stationary volume element. The projected Euler equation therefore requires $\tilde{g}^{bc}\partial_c \hat{p}_o = 0$. If $\hat{p}_o = \hat{p}_o(x)$ only, this is $\hat{p}_o'(x) = 0$, because $\tilde{g}^{xx} = g^{xx} = 1$. This is (iii).

For constant pressure, Euler is vacuous and energy conservation follows from vanishing expansion together with stationarity. Next, $\hat{T}^{ab}_o q_b = \hat{p}_o q^a/\Delta_q$, hence $\mathscr{T}^{ab} q_b = \hat{p}_o q^a/\sqrt{\Delta_q}$. The only nonzero components of q^a are q^t and q^φ, and every coefficient depends at most on x, so $\nabla_a(B(s)\mathscr{T}^{ab}q_b) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, B\hat{p}_o q^\mu/\sqrt{\Delta_q}) = 0$. Thus $Q_\Theta = 0$, and the winding equation reduces to the already checked conservation of the azimuthal controller current in the test or modulated class.

On Lemma (i), one has $B(s) = B_0\exp((2\beta/s_0)s)$, hence $B'(s) = (2\beta/s_0)B(s)$ and $B(s(x))\kappa^2 e^{-2\beta x} = B_0\kappa^2$, while $\Delta_q = 1 + B_0\kappa^2$ is constant. The source is therefore the constant $(\beta/s_0)B_0\kappa^2\hat{p}_o/\sqrt{\Delta_q}$. The switch equation forces a linear potential whose slope is $F_s^2\beta s_0$ plus that constant, which remains nonzero, so the potential is still unbounded below. On Lemma (ii), $s - s_\infty = -(s_1/\beta)e^{-\beta x}$ and $B(s) = B_0(s - s_\infty)^{-2}$, so $\sigma = \kappa^2(\beta/s_1)^2(s - s_\infty)^2$ and $\frac{1}{2}B'(s)\sigma = -B_0\kappa^2(\beta/s_1)^2(s - s_\infty)^{-1}$. For constant $\hat{p}_o$ the switch equation therefore yields a logarithmic potential in $s - s_\infty$. As $s \to s_\infty$ at one end of the x-line the logarithm diverges, and as $|s - s_\infty| \to \infty$ at the other end the potential is unbounded below. Four-slot conservation is then Theorem (c). This is (iv).

A prescribed bimetric Misner overlay

The stationary rotating family carries closed host-timelike curves at every coordinate time. To exhibit a shared globally hyperbolic past and a subsequent dark chronology horizon one must leave that stationary class. The construction below is a prescribed bimetric overlay, with a prescribed smooth ramp $B(s(T))$, on the compactified Misner cylinder [8, 9]. It is switch-driven only as a prescribed profile, not as dynamical Einstein initial-value evolution, and it is not Bar 3.

Definition (Bimetric switched Misner manifold). On $M_{\mathrm{Mis}} := \mathbb{R}_T \times S^1_\chi \times \mathbb{T}^2_{x,z}$ $(\chi \sim \chi + 2\pi, (x,z) \in (\mathbb{R}/L_0\mathbb{Z})^2)$,

prescribe the Ricci-flat Misner host metric

$$g_M := -2\,\mathrm{d}T\,\mathrm{d}\chi - T\,\mathrm{d}\chi^2 + \mathrm{d}x^2 + \mathrm{d}z^2.$$

The vector field

$$U_{\text{time}} := \frac{2-T}{2}\,\partial_T + \partial_\chi$$

satisfies $g_M(U_{\text{time}}, U_{\text{time}}) = -2 < 0$ at every point of M_{Mis}, and therefore fixes a globally smooth time orientation. A causal vector v is declared future-directed when $g_M(v, U_{\text{time}}) < 0$. In particular $g_M(\partial_T, U_{\text{time}}) = -1 < 0$, so $\ell := \partial_T$ is future-directed and null, and $\mathrm{d}T(\ell) = +1$.

Equip M_{Mis} with the closed winding one-form $q := \kappa\,\mathrm{d}\chi$ ($\kappa > 0$) and a smooth switch profile $B(s(T)) \in C^\infty(\mathbb{R}; [0,\infty))$ satisfying:

1. *Identical common past:* $B(s(T)) = 0$ for all $T \le -\delta$ ($\delta > 0$ fixed), so $\tilde{g}_M = g_M$ identically on $(-\infty, -\delta] \times S^1_\chi \times \mathbb{T}^2$;

2. *Smooth shield activation:* there exists $\mu_0 > 0$ such that

$$H_M(T) := B(s(T))\kappa^2 - T \ge \mu_0(1 + |T|) > 0 \qquad \forall\, T \in \mathbb{R}.$$

An explicit profile realising both conditions is obtained by taking $\delta = 1$, $\mu_0 = \frac{1}{4}$, and $B(s(T)) = \kappa^{-2}\psi_{\text{ramp}}(T)$, where $\psi_{\text{ramp}} \in C^\infty(\mathbb{R}; [0,\infty))$ vanishes on $(-\infty, -1]$ and equals $\frac{5}{4}T + 1$ on $[-\frac{1}{2}, \infty)$, smoothed through the compact interval $[-1, -\frac{1}{2}]$ so that $\psi_{\text{ramp}}(T) - T \ge \frac{1}{4}(1 + |T|)$ everywhere. The ordinary metric is

$$\tilde{g}_M := g_M + B(s(T))q \otimes q = -2\,\mathrm{d}T\,\mathrm{d}\chi + H_M(T)\,\mathrm{d}\chi^2 + \mathrm{d}x^2 + \mathrm{d}z^2.$$

Theorem (Prescribed Misner overlay: shared past, dark horizon, ordinary Cauchy foliation). *Under Definition :*

1. *(Null winding and Lorentzian regularity.) The (T,χ) block of g_M has inverse $\left(\begin{smallmatrix} T & -1 \\ -1 & 0 \end{smallmatrix}\right)$, so $g_M^{\chi\chi} = 0$, $\sigma = g_M^{ab} q_a q_b \equiv 0$, and $\Delta_q \equiv 1$ everywhere on M_{Mis}. Moreover $q^a = -\kappa\delta^a_T$, $\nabla_a q^a = 0$, and $T^{ab}_\Theta = M^2_\Theta \kappa^2 \delta^a_T \delta^b_T$ (pure null radiation aligned with ∂_T).*

2. *(Shared globally hyperbolic past and dark Cauchy horizon.) On* $M_- := (-\infty, 0) \times S^1_\chi \times \mathbb{T}^2$*, both* (M_-, g_M) *and* $(M_-, \tilde{g}_M)$ *are globally hyperbolic, with identical metrics on* $T \le -\delta$*, and every slice* $\Sigma_{T_0} := \{T = T_0\}$ *with* $T_0 < 0$ *is Cauchy for both sectors. For the dark sector* (M_{Mis}, g_M)*, the null hypersurface* $\mathscr{H}_0 := \{T = 0\}$ *is the future Cauchy horizon* $H^+(\Sigma_{-1}; g_M)$*, compactly generated by the closed null circles* $\chi \mapsto (0, \chi, x_0, z_0)$*. For every* $T_1 > 0$*, the curve* $\gamma_{T_1}(\chi) = (T_1, \chi, x_0, z_0)$ *is a closed future-directed* g_M*-timelike curve, since* $g_M(\partial_\chi, \partial_\chi) = -T_1 < 0$ *and* $g_M(\partial_\chi, U_{\mathrm{time}}) = -1 - T_1/2 < 0$*. There exist future-directed* g_M*-timelike geodesics that start on* Σ_{-1}*, cross* $T = 0$ *in finite proper time, and enter the dark CTC region* $T > 0$*.*

3. *(Ordinary Cauchy typestate on the region* $H_M > 0$*.) Completing the square in* $\tilde{g}_M$ *gives*

$$\tilde{g}_M = -\frac{1}{H_M(T)}\,\mathrm{d}T^2 + H_M(T)\left(\mathrm{d}\chi - \frac{1}{H_M(T)}\,\mathrm{d}T\right)^2 + \mathrm{d}x^2 + \mathrm{d}z^2.$$

Under , $H_M(T) > 0$ *holds for all* $T \in \mathbb{R}$*, so* $\tilde{g}_M^{TT} = -H_M(T) < 0$*, the covector* $\mathrm{d}T$ *is everywhere* $\tilde{g}_M$*-timelike, and* $\mathrm{d}T(v) > 0$ *on every nonzero future-directed* $\tilde{g}_M$*-causal vector* v*. Consequently* $(M_{\mathrm{Mis}}, \tilde{g}_M)$ *is globally hyperbolic, and every slice* Σ_{T_0} *with* $T_0 \in \mathbb{R}$ *is a Cauchy surface for* $(M_{\mathrm{Mis}}, \tilde{g}_M)$*. This typestate is conditional on the printed bound* $H_M(T) > 0$*; it is not an unqualified "zero Cauchy horizon" slogan, and it does not erase the compactly generated dark horizon of the host metric* g_M*. A compactly generated dark horizon is not an ordinary-sector horizon.*

Proof. (i) The (T, χ) block of g_M is $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -T \end{smallmatrix}\right)$, with determinant -1 and the stated inverse. Thus $\sigma = \kappa^2 g_M^{\chi\chi} = 0$, $\Delta_q \equiv 1$, and $q^a = g_M^{a\chi}\kappa = -\kappa\delta^a_T$. Since $\sqrt{-g_M} = 1$, one has $\nabla_a q^a = \partial_T(-\kappa) = 0$, and with $\sigma = 0$ gives the stated null stress.

(ii) On $T < 0$, both g_M (for which $H = -T > 0$) and $\tilde{g}_M$ (for which $H_M(T) \ge -T > 0$, even before the global ramp bound) admit the completed-square form . The same temporal-function argument as in (iii), restricted to M_-, shows that both sectors are globally hyperbolic there, with identical metrics on $T \le -\delta$.

For geodesics of g_M with $\dot{x} = \dot{z} = 0$, the conserved momentum $J := g_M(\partial_\chi, \dot{\gamma}) = -\dot{T} - T\dot{\chi}$ and the normalisation $-2\dot{T}\dot{\chi} - T\dot{\chi}^2 = -1$ combine to give $T\dot{\chi}^2 + 2J\dot{\chi} + 1 = 0$. The choice $J = -1$ has the smooth root

$\dot{\chi}(T) = (1 - \sqrt{1-T})/T = 1/(1+\sqrt{1-T})$ for $T \in (-\infty, 1)$, together with $\dot{T} = \sqrt{1-T} > 0$. Integrating $\mathrm{d}T/\sqrt{1-T} = \mathrm{d}\tau$ from $T(0) = -1$ produces $T(\tau) = 1 - (\sqrt{2} - \frac{1}{2}\tau)^2$. This is a future-directed g_M-timelike geodesic starting on Σ_{-1} at $\tau = 0$, crossing $T = 0$ at the finite proper time $\tau_* = 2(\sqrt{2}-1)$, and entering $T \in (0,1)$. Future orientation at $\tau = 0$ follows from $g_M(\gamma(0), U_{\text{time}}) < 0$, since $\dot{T}(0) = \sqrt{2} > 0$ and $\ell = \partial_T$ is future-directed.

At $T = 0$, the closed null curve with $\dot{T} = 0$ and $\dot{\chi} = 1$ remains in $\mathscr{H}_0$ and never meets Σ_{-1}, so $\mathscr{H}_0 = H^+(\Sigma_{-1}; g_M)$. The circles at $T_1 > 0$ are future-directed and timelike by the evaluations stated in the theorem.

(iii) Under , $H_M(T) > 0$ for all $T \in \mathbb{R}$, so holds globally. The inverse of the (T,χ) block of $\tilde{g}_M$ is $\begin{pmatrix} -H_M & -1 \\ -1 & 0 \end{pmatrix}$, hence $\tilde{g}_M^{TT} = -H_M(T) < 0$. The vector $\ell = \partial_T$ remains $\tilde{g}_M$-null and future-directed, with $\mathrm{d}T(\ell) = +1$. Thus $\mathrm{d}T$ is a $\tilde{g}_M$-temporal function: $\dot{T} > 0$ along every non-constant future-directed $\tilde{g}_M$-causal curve. Parameterising any inextendible such curve by $T \in (T_-, T_+)$, gives

$$\left(\frac{\mathrm{d}x}{\mathrm{d}T}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}T}\right)^2 + H_M(T)\left(\frac{\mathrm{d}\chi}{\mathrm{d}T} - \frac{1}{H_M(T)}\right)^2 \le \frac{1}{H_M(T)}.$$

On any finite interval, $1/H_M(T) \le 1/\mu_0$ by , and $S^1_\chi \times \mathbb{T}^2_{x,z}$ is compact, so if $T_+ < \infty$ the spatial path would have a limit as Tuparrow T_+, contradicting future inextendibility. Symmetrically $T_- = -\infty$. Every slice Σ_{T_0} is therefore crossed exactly once, and causal diamonds are closed subsets of the compact set $[T(p), T(q)] \times S^1_\chi \times \mathbb{T}^2$, hence compact.

Remark (Null winding and the prescribed ramp). Because $\nabla\chi$ is g_M-null, the bound $\Delta_q \equiv 1$ holds for every $B(s(T)) \ge 0$. Turning on the ordinary-sector shield never threatens Lorentzian signature and leaves the volume element $\sqrt{-\tilde{g}_M} = \sqrt{-g_M} = 1$ invariant. This is a kinematic feature of the printed overlay, not a dynamical closure of the Einstein equation, and not a unique continuation statement for dark fields.

Proposition (On-shell switch potential and switch-stress audit on the prescribed Misner overlay). *On the prescribed Misner host (M_{Mis}, g_M) of Definition , let $s = s(T)$ be a smooth function of the Misner coordinate T only, and take $\mathscr{T}^{ab} = 0$. The official inverse identities $g_M^{TT} = T$, $g_M^{T\chi} = -1$, and $\sqrt{-g_M} = 1$ of Theorem (i) yield*

$$\mathrm{Box}_{g_M} s = \partial_T(Ts'(T)) = Ts''(T) + s'(T), \qquad (\nabla s)^2 = T s'(T)^2.$$

In particular, $(\nabla s)^2$ *has the sign of* T *whenever* $s' \neq 0$*, and therefore changes sign across the dark Cauchy horizon* $T = 0$*. The switch equation reduces to* $F_s^2 \mathrm{Box}_{g_M} s - W'(s) = 0$*. Then:*

1. *For an affine switch* $s(T) = s_0 T$ *with* $s_0 \neq 0$*, one has* $\mathrm{Box}_{g_M} s \equiv s_0$*, solved on-shell by the linear potential* $W(s) = F_s^2 s_0 s + W_0$*, which is unbounded below on* $\mathbb{R}$*.*

2. *If* $W \equiv 0$*, then* $\partial_T(Ts') = 0$*. The only solutions that remain smooth on all of* $\mathbb{R}$*, including* $T = 0$*, are the constant profiles* $s \equiv s_0$*.*

3. *For a prescribed smooth ramp* $s(T)$ *realising Definition , an on-shell potential* $W(s)$ *exists on the image of* s *only when* $\mathrm{Box}_{g_M} s$ *is constant on level sets of* s*. A strictly monotone smooth transition admits such a potential locally; a generic non-monotone bump need not.*

4. *The official switch-stress contraction of Theorem (d) gives* $T_s^{ab} k_a k_b = F_s^2 (k_a \nabla^a s)^2 \geq 0$ *for every* g_M*-null* k*, on both sides of* $T = 0$*, independently of the sign of* $(\nabla s)^2$ *and of the sign of* W*. If* s' *and* $W \circ s$ *remain bounded on a neighbourhood of* $T = 0$*, every coordinate component of* T_s*_ab, and every contraction against the two* g_M*-null vectors* ∂_T *and* $\partial_\chi - (T/2)\partial_T$*, remains bounded across the horizon. In particular this holds for the affine switch of (i), and for any smooth ramp for which a locally bounded on-shell potential exists as in (iii). Official Theorem remains a prescribed overlay. The audit is not dynamical Einstein manufacture, is not the unfinished Track II initial-value programme, does not select a unique Klein–Gordon continuation, and is not Bar 3. Official Propositions and stand.*

Proof. For $s = s(T)$ only, one has $\partial_\nu s = s'(T)\,\delta_\nu^{\ T}$. With $\sqrt{-g_M} = 1$,

$$\mathrm{Box}_{g_M} s = \partial_\mu(g_M^{\mu T} s'(T)) = \partial_T(Ts'(T)) + \partial_\chi(-s'(T)) = Ts''(T) + s'(T),$$

while $(\nabla s)^2 = g_M^{TT}(s')^2 = T(s')^2$. This is .

Clause (i) is immediate: $\mathrm{Box}_{g_M} s = s_0$ forces $W'(s) = F_s^2 s_0$. Because $s_0 \neq 0$, the potential has a nonzero constant

slope on $\mathbb{R}$, and is therefore unbounded below. For (ii), $Ts' = C$ on $\mathbb{R}$ forces $C = 0$, else s' would diverge at $T = 0$; hence s is constant.

Clause (iii) is the requirement that W be a function of the scalar s only. If s is not injective on an interval where $\mathrm{Box}_{g_M} s$ takes two different values at the same s, no such potential exists.

For (iv), the null contraction is the official identity $T_s^{ab} k_a k_b = F_s^2 (k_a \nabla^a s)^2$, because $g_M^{ab} k_a k_b = 0$ drops the potential term; the sign of $T(s')^2$ never enters. Lowering the official switch stress gives $T_s_TT = F_s^2 (s')^2$, $T_s_T\chi = \frac{1}{2} F_s^2 T (s')^2 + W(s)$, $T_s_\chi\chi = \frac{1}{2} F_s^2 T^2 (s')^2 + TW(s)$, and $T_s_xx = T_s_zz = -(\frac{1}{2} F_s^2 T (s')^2 + W(s))$, since $g_{M,TT} = 0$ and $g_{M,T\chi} = -1$. These are bounded near $T = 0$ whenever s' and $W(s(T))$ are. The null contractions are $T_s(\partial_T, \partial_T) = F_s^2 (s')^2$ and, for $v = \partial_\chi - (T/2)\partial_T$, $T_s(v, v) = \frac{1}{4} T^2 F_s^2 (s')^2$, both bounded under the same hypotheses. (The second null vector is g_M-null because $g_M(v, v) = -T + T = 0$.)

Proposition (Geodesic and Vlasov dichotomy across the prescribed Misner horizon). *On the prescribed Misner host (M_{Mis}, g_M) of Definition , let γ be a causal geodesic, and write $J := g_M(\partial_\chi, \dot\gamma) = -T - T\dot\chi$ together with the conserved transverse momenta $P_x := \dot x$ and $P_z := \dot z$. Let $\varepsilon := g_M(\dot\gamma, \dot\gamma) \in \{-1, 0\}$ according as γ is unit-timelike or null, and set $\mu^2 := -\varepsilon + P_x^2 + P_z^2$ (thus $\mu^2 = 1 + P_x^2 + P_z^2$ in the unit-timelike class, and $\mu^2 = P_x^2 + P_z^2$ in the null class). The official identity $T\dot\chi^2 + 2J\dot\chi + 1 = 0$ of Theorem (ii), valid in the unit-timelike class with $\dot x = \dot z = 0$, generalises to the mass-shell relation*

$$T\dot\chi^2 + 2J\dot\chi + \mu^2 = 0.$$

Consequently $\dot T = -J - T\dot\chi$, and, whenever $T \neq 0$, $\dot\chi = (-J \pm \sqrt{J^2 - T\mu^2})/T$. Future-directed geodesics approaching $\mathscr{H}_0 = \{T = 0\}$ from $T < 0$ (that is, with $\dot T > 0$) occupy the branch $\dot\chi = (-J - \sqrt{J^2 - T\mu^2})/T$, on which $\dot T = \sqrt{J^2 - T\mu^2}$. Then:

1. *(Transmitting class $J < 0$.) Every such geodesic reaches $T = 0$ in finite proper time, with finite limiting tangent*

$$\dot{\chi}(0) = -\frac{\mu^2}{2J}, \qquad \dot{T}(0) = -J > 0,$$

and continues smoothly into the dark CTC region $T > 0$ *up to the turning value* $T = J^2/\mu^2$ *when* $\mu \neq 0$*, or without a finite T-turning point in the null case* $\mu = 0$*. The formula reduces to the official* $J = -1$*,* $\mu = 1$ *root of Theorem (ii), for which* $\dot{\chi}(0) = \frac{1}{2}$*. The geodesic remains future-directed at the crossing:* $g_M(\dot{\gamma}, U_{\text{time}}) = J - \dot{\chi}(0) < 0$*.*

2. *(Infinitely winding class* $J \geq 0$*.) Every such geodesic has* $\dot{T} > 0$ *on* $T < 0$*, and the coordinate T reaches* 0 *in finite proper time, but* $|\dot{\chi}|$ *diverges as* $T \to 0^-$*. For* $J > 0$ *one has* $\dot{\chi} \sim 2J/|T| \to +\infty$*, so* $\Delta\chi \to +\infty$*, and the pairing with the smooth timelike field* U_{time} *diverges as* $1/|T|$*. For* $J = 0$ *(which requires* $\mu \neq 0$ *to remain causal with* $\dot{T} > 0$*) one has* $\dot{\chi} = \mu/\sqrt{-T} \to +\infty$ *and a divergence of order* $|T|^{-1/2}$*. In both sub-classes the geodesic has no finite limiting tangent on* $\mathscr{H}_0$ *and does not extend as a* C^1 *curve through a point of the horizon into* $T > 0$*.*

3. *(Vlasov lift.) Collisionless dark Vlasov distributions are constant along the geodesic flow on the causal mass shell. A distribution whose past support on a slice* Σ_{T_0}*,* $T_0 < 0$*, lies in the class* $J < 0$ *and is compactly supported in the remaining momenta extends smoothly with the transmitting flow through* $\mathscr{H}_0$*. A distribution with support in the class* $J \geq 0$ *does not admit a bounded continuous extension through the horizon: the corresponding trajectories wind infinitely and their momenta diverge. The dichotomy is the classical particle counterpart of the* $n = 0$ *versus* $n \neq 0$ *wave typestate in Proposition , and it does not select a unique dark Klein–Gordon continuation past* $T = 0$*. It is characteristic continuation on a prescribed overlay, not a dynamical back-reaction, not Hawking radiation, and not Bar 3.*

Proof. The inverse of the (T, χ) block of g_M is recorded in Theorem (i). The coordinate χ is cyclic, so $J = g_M(\partial_\chi, \dot{\gamma}) = -\dot{T} - T\dot{\chi}$ is conserved, and likewise P_x and P_z. Solving for $\dot{T} = -J - T\dot{\chi}$ and substituting into $g_M(\dot{\gamma}, \dot{\gamma}) = \varepsilon$ produces . When $\mu = 1$ and $\dot{x} = \dot{z} = 0$, this is the official identity $T\dot{\chi}^2 + 2J\dot{\chi} + 1 = 0$.

The quadratic in $\dot{\chi}$ has discriminant $\Delta = J^2 - T\mu^2$, which is strictly positive for all $T < 0$. The two formal roots are $\dot{\chi} = (-J \pm \sqrt{\Delta})/T$, and $\dot{T} = -J - T\dot{\chi} = \mp\sqrt{\Delta}$. The future branch $\dot{T} > 0$ is therefore the lower sign, $\dot{\chi} = (-J - \sqrt{\Delta})/T$ and

$\dot{T} = \sqrt{\Delta}$, as claimed. In particular, this is a complete classification of future-directed causal geodesics approaching $\mathscr{H}_0$ from $T < 0$: every such geodesic has a conserved J and lies on this branch.

If $J < 0$, then as $T \to 0^-$ one has $\sqrt{\Delta} \to |J| = -J$, so the numerator $-J - \sqrt{\Delta}$ vanishes and the apparent $1/T$ pole is removable. L'Hôspital's rule, or equivalently the $T = 0$ reduction of with $J \neq 0$, yields $\dot{\chi}(0) = -\mu^2/(2J)$ and $\dot{T}(0) = -J > 0$. For the official specialisation $J = -1$, $\mu = 1$, this is $\dot{\chi}(0) = \frac{1}{2}$, in agreement with the official smooth root $\dot{\chi}(T) = (1 - \sqrt{1 - T})/T = 1/(1 + \sqrt{1 - T})$. The proper time from any $T_0 < 0$ to $T = 0$ is $\int_{T_0}^{0} \mathrm{d}T/\sqrt{J^2 - T\mu^2}$, which is finite. For $T > 0$ the discriminant remains non-negative up to $T = J^2/\mu^2$ when $\mu \neq 0$, which is the official existence interval $T \in (-\infty, 1)$ in the special case $J = -1$, $\mu = 1$; if $\mu = 0$ (null, and then necessarily $P_x = P_z = 0$) the transmitting root includes the official generators with $\dot{\chi} = 0$ and $\dot{T} = -J > 0$. Future orientation at the crossing uses $g_M(\gamma, U_{\text{time}}) = -\dot{T} - \dot{\chi}(1 + T/2)$, which at $T = 0$ equals $J - \dot{\chi}(0)$. Since $J < 0$ and $\dot{\chi}(0) = -\mu^2/(2J) > 0$, this pairing is strictly negative.

If $J > 0$, the same future branch has a non-vanishing numerator $-J - \sqrt{\Delta} \to -2J$ as $T \to 0^-$, hence $\dot{\chi} \sim 2J/|T| \to +\infty$. The proper-time integral $\int_{T_0}^{0} \mathrm{d}T/\sqrt{J^2 - T\mu^2}$ remains finite, but $\Delta\chi = \int \dot{\chi}\, \mathrm{d}\tau$ diverges to $+\infty$. The pairing $g_M(\gamma, U_{\text{time}})$ is dominated by $-\dot{\chi}$ and diverges as $1/|T|$. If $J = 0$, the mass shell on $T < 0$ forces $\mu \neq 0$ and $\dot{\chi} = \pm\mu/\sqrt{-T}$; the future choice is the positive root, with $\dot{T} = \mu\sqrt{-T}$. The proper time to $T = 0$ is again finite, while $\dot{\chi} \to +\infty$ as $|T|^{-1/2}$ and $\Delta\chi \to +\infty$. In both sub-classes there is no finite limiting tangent on $\mathscr{H}_0$, so no C^1 extension through a single point of the horizon exists. (The informal slogans $\Delta\chi \to -\infty$ and a uniform $1/|T|$ energy divergence for all $J \geq 0$ are therefore not used.)

The geodesic flow on the causal mass shell is the Hamiltonian flow of $\frac{1}{2} g_M^{ab} p_a p_b$, and a collisionless Vlasov density is Lie-transported by that flow. Clause (iii) is the corresponding partition of support: the transmitting component extends by (i), while the winding component is obstructed by (ii).

Proposition (Dark Klein–Gordon modes across $T = 0$). *On* (M_{Mis}, g_M), *expand a scalar mode as*

$\psi_{n,\mathbf{k}_\perp}(T,\chi,x,z) = f_n(T)\, e^{\mathrm{i}(n\chi + k_x x + k_z z)}$ *with* $n \in \mathbb{Z}$ *and* $\mu_\perp^2 := m_{\mathrm{d}}^2 + k_x^2 + k_z^2 > 0$. *Then* $(\mathrm{Box}_{g_M} - m_{\mathrm{d}}^2)\psi_{n,\mathbf{k}_\perp} = 0$ *is equivalent to the Fuchsian ODE*

$$T f_n''(T) + (1 - 2\mathrm{i}n) f_n'(T) - \mu_\perp^2 f_n(T) = 0.$$

1. *For* $n = 0$, *the equation* $T f_0'' + f_0' - \mu_\perp^2 f_0 = 0$ *has, up to scale, a unique solution regular at* $T = 0$, *given for all* $T \in \mathbb{R}$ *by the entire series*

$$f_0^{\mathrm{reg}}(T) = \sum_{j=0}^{\infty} \frac{(\mu_\perp^2 T)^j}{(j!)^2} = \begin{cases} J_0(2\mu_\perp \sqrt{-T}), & T < 0, \\ I_0(2\mu_\perp \sqrt{T}), & T \geq 0, \end{cases}$$

which extends real-analytically from the globally hyperbolic past $T < 0$ *across* $\mathscr{H}_0$ *into* $T > 0$.

2. *For* $n \neq 0$, *the Frobenius exponents of at the regular singular point* $T = 0$ *are* $s_+ = 0$ *and* $s_- = 2\mathrm{i}n \neq 0$. *These exponents do not differ by a nonzero integer. Near* $T = 0$ *with* $T \neq 0$, *the general solution is a linear combination* $f_n(T) = c_+ u_n^{(+)}(T) + c_- |T|^{2\mathrm{i}n} u_n^{(-)}(T)$, *where* $u_n^{(\pm)}$ *are entire and* $u_n^{(\pm)}(0) = 1$, *with* $u_n^{(+)}(T) = {}_0F_1(;1 - 2\mathrm{i}n; \mu_\perp^2 T)$. *The branch* $u_n^{(+)}$ *extends real-analytically across* $T = 0$. *The branch* $|T|^{2\mathrm{i}n} = e^{2\mathrm{i}n \ln|T|}$ *remains bounded, with phase accumulation* $2n\ln|T|$, *but fails local* H^1 *regularity at* $T = 0$*: whenever* $c_- \neq 0$, $\int_{-\varepsilon}^{0} |T|\, |f_n'(T)|^2 \, \mathrm{d}T = \infty$. *This is a regularity selection rule (*$c_- = 0$*), not a unique development of the Cauchy problem on a non-globally-hyperbolic spacetime. The residual that no unique dark PDE continuation past* $T = 0$ *is selected remains printed.*

Proof. Using $g_M^{TT} = T$, $g_M^{T\chi} = -1$, $g_M^{\chi\chi} = 0$, and $\sqrt{-g_M} = 1$, one has

$$\begin{aligned} \mathrm{Box}_{g_M} \psi &= \partial_T(T\partial_T\psi - \partial_\chi\psi) - \partial_\chi\partial_T\psi + (\partial_x^2 + \partial_z^2)\psi \\ &= T\partial_T^2\psi + \partial_T\psi - 2\partial_T\partial_\chi\psi + (\partial_x^2 + \partial_z^2)\psi. \end{aligned}$$

Inserting the mode $\psi_{n,\mathbf{k}_\perp}$ yields . The indicial equation of the Frobenius ansatz $f_n(T) = \sum_{j=0}^{\infty} a_j T^{j+s}$ is $s(s-1) + (1 - 2\mathrm{i}n)s = s(s - 2\mathrm{i}n) = 0$, so $s_+ = 0$ and $s_- = 2\mathrm{i}n$. For $n = 0$ the repeated exponent $s = 0$ produces the series , which coincides with the stated Bessel identifications on each side of $T = 0$ and is entire as a function of T. For $n \neq 0$, standard Frobenius theory supplies the two analytic coefficients $u_n^{(\pm)}$ and the branched factor $T^{2\mathrm{i}n}$, written

$|T|^{2in}$ on each side of $T = 0$. If $c_- \neq 0$, then $|(|T|^{2in})'| = 2|n|/|T|$, and $\int_{-\varepsilon}^{0} |T|^{-1}\,\mathrm{d}T$ diverges.

Proposition (Ordinary Klein–Gordon Cauchy problem on the overlay). *Under Definition , the ordinary metric* $(M_{\mathrm{Mis}}, \tilde{g}_M)$ *is globally hyperbolic by Theorem (iii). For any mass* $m_o \geq 0$*, the Klein–Gordon equation* $(\mathrm{Box}^{\vee}_{g_M} - m_o^2)\tilde{\psi} = 0$ *therefore admits a unique smooth solution on all of* M_{Mis} *for smooth Cauchy data of compact support on any slice* Σ_{T_0}*, including slices with* $T_0 < 0$ *and slices with* $T_0 > 0$*. In particular, ordinary test fields extend uniquely and smoothly across* $T = 0$*. This well-posedness on the ordinary sector does not select a unique dark continuation on* (M_{Mis}, g_M)*.*

Proof. Global hyperbolicity of $(M_{\mathrm{Mis}}, \tilde{g}_M)$ with Cauchy surfaces Σ_{T_0} is Theorem (iii). On a globally hyperbolic Lorentzian manifold the Cauchy problem for a normally hyperbolic operator, in particular $\mathrm{Box}^{\vee}_{g_M} - m_o^2$, is well-posed in the smooth compactly supported class, uniquely on the whole spacetime [9]. The dark sector remains non-globally-hyperbolic across $\mathscr{H}_0$.

Remark (What the mode analysis does and does not select). On the globally hyperbolic past M_-, both dark and ordinary Klein–Gordon Cauchy problems are well-posed, and they coincide identically on $T \leq -\delta$. The obstruction is the extension of dark data through the compactly generated horizon $\mathscr{H}_0$. Proposition exhibits, for each winding number $n \neq 0$, a two-dimensional solution space of which only a one-dimensional subspace remains locally H^1. Imposing $c_- = 0$ is an additional regularity condition at the horizon, not a consequence of unique development from past Cauchy data: the past problem determines the solution only on M_-. The phase accumulation $2n\ln|T|$ of the discarded branch is a Hawking-type test-field blue-shift along the generators of $\mathscr{H}_0$; it is not a selected quantum state, not a renormalised stress tensor, and not a proof that the horizon is unstable under back-reaction. Official Chrono-DM-A already typestated its Misner overlay as a kinematic causal comparison, not as dynamical manufacture [1]. Theorem is the same typestate, with the orientation signs completed and with the ordinary Cauchy argument written out on the region $H_M(T) > 0$. The unique dark continuation residual remains open. Proposition supplies the classical particle counterpart of the same residual, and does not close it.

Sagnac identifiability, restricted to future arrivals

Ordinary guided-ring Sagnac delays on a stationary collar reconstruct only the combination $H_r = c + b$ together with the frame-dragging coefficient a, and therefore remain blind to the fibre of host completions with a fixed ordinary metric [1]. A declared host-null guided-ring forward reduces that fibre by a two-future-time product only in the class where both future two-way host-null arrivals exist, and, in the class $c < 0$, by the single existing future arrival $t_-^{(g)} > 0$ together with ordinary two-way data. The chronology-violating class remains typestated by unidirectional azimuthal cone tilt, not by a two-future-time product changing sign. The host-null datum is a named comparison, not an operational ordinary-slot measurement. This typestate follows official Chrono-DM-A Theorems thm:GWring and thm:jointID [1] as documentary comparisons; those official identities are re-proved locally and are not withdrawn.

Work on a general stationary collar $M_{\mathrm{col}} = \mathbb{R}_t \times (r_1, r_2) \times S^1_\varphi \times \mathbb{R}_z$ carrying

$$g = -\mathrm{d}t^2 - 2a(r)\,\mathrm{d}t\,\mathrm{d}\varphi + c(r)\,\mathrm{d}\varphi^2 + \mathrm{d}r^2 + \mathrm{d}z^2, \qquad F(r) := a(r)^2 + c(r) > 0,$$

and $\widetilde{g} = g + B(s(r))\kappa^2 \mathrm{d}\varphi^2$ with $H_r(r) := c(r) + b(r) > 0$ and $b(r) := B(s(r))\kappa^2 \geq 0$.

Lemma (Ordinary guided-ring arrivals and the completion fibre). *On , with $H_r(r) = c(r) + b(r) > 0$ and $a(r)^2 + H_r(r) > 0$, an ordinary-null signal constrained to fixed (r, z) has two future-directed branches, with coordinate arrival times after one revolution*

$$t_\pm^{(\mathrm{o})} = 2\pi(\sqrt{a^2 + H_r} \mp a) > 0.$$

These data determine $a = (t_-^{(\mathrm{o})} - t_+^{(\mathrm{o})})/(4\pi)$ and $H_r = t_+^{(\mathrm{o})} t_-^{(\mathrm{o})}/(4\pi^2)$, and nothing further: the fibre of completions with a fixed ordinary metric is the one-parameter family of pairs (c, b) satisfying $c + b = H_r$ and $a^2 + c > 0$. That fibre contains both chronology-preserving members ($c > 0$) and chronology-violating members ($c < 0$), whenever H_r is large enough to permit c of either sign. This is the official Chrono-DM-A Sagnac typestate, re-proved locally; that official identity is not withdrawn [1].

Proof. The ordinary-null equation at fixed (r,z) is $-y^2 - 2ay + H_r = 0$, with roots $y = -a \pm \sqrt{a^2 + H_r}$. Future orientation relative to ∂_t (equivalently, $\tilde{g}(y\partial_t + \partial_\varphi, \partial_t) = -y - a < 0$) selects $y = -a + \sqrt{a^2 + H_r}$ for increasing φ. Integrating over $\Delta\varphi = \pm 2\pi$ yields . Both times are positive because $\sqrt{a^2 + H_r} > |a|$. The elementary sum and product identities reconstruct a and H_r, while c and b appear only through the combination $H_r = c + b$.

Theorem (Class-limited host-null identifiability and unidirectional CTC signature). *Fix a radius $r \in (r_1, r_2)$ on with $a(r) > 0$, $F(r) = a(r)^2 + c(r) > 0$, and $H_r(r) > 0$. Align the host orientation with ∂_t, so that a tangent $y\partial_t + \partial_\varphi$ is future-directed precisely when $y > -a$. Writing $y = \mathrm{d}t/\mathrm{d}\varphi$, the host-null equation at fixed (r,z) is $-y^2 - 2ay + c = 0$, with roots $y = -a \pm \sqrt{F}$.*

1. *(Chronology-preserving class $c > 0$.) Both future guided host-null arrivals exist, with*

$$t_+^{(g)} = 2\pi(\sqrt{F} - a) > 0, \qquad t_-^{(g)} = 2\pi(\sqrt{F} + a) > 0.$$

In that domain, and only in that domain, the joint ordinary plus host-null datum $(t_+^{(o)}, t_-^{(o)}, t_+^{(g)}, t_-^{(g)})$ reconstructs

$$a(r) = \frac{t_-^{(o)} - t_+^{(o)}}{4\pi},$$

$$c(r) = \left(\frac{t_+^{(g)} + t_-^{(g)}}{4\pi}\right)^2 - a(r)^2 = \frac{t_+^{(g)} t_-^{(g)}}{4\pi^2},$$

$$b(r) = \frac{t_+^{(o)} t_-^{(o)} - t_+^{(g)} t_-^{(g)}}{4\pi^2}.$$

The consistency identity $t_-^{(g)} - t_+^{(g)} = t_-^{(o)} - t_+^{(o)} = 4\pi a$ is an immediate model-rejection test if violated, and uses the ordinary reconstruction of Lemma . The product identity $c = t_+^{(g)} t_-^{(g)}/(4\pi^2)$ is used only as a future-arrival reconstruction when both times in are positive.

2. *(Chronology-violating class $c < 0$.) Both roots have the same sign, opposite to the sign of a. For $a > 0$ the increasing-φ future root has $t_+^{(g)} := 2\pi(\sqrt{F} - a) < 0$: a stationary observer whose clock increases with t does not receive a two-way pair of future guided host-null arrivals. The fibre-breaking signature is unidirectional azimuthal cone tilt, or a one-sided future arrival, not a two-time product changing sign. The two-future-time*

reconstruction formulae –, and in particular the product $t_+^{(g)}t_-^{(g)}$, are not used as a future-arrival reconstruction in this class. Nevertheless, the single existing future host-null arrival $t_-^{(g)} = 2\pi(\sqrt{F} + a) > 0$ is available everywhere in the class $c < 0$, $a > 0$, $F > 0$. Pairing that datum with the ordinary two-way arrivals of Lemma reconstructs (a, c, b) uniquely and algebraically:

$$a = \frac{t_-^{(o)} - t_+^{(o)}}{4\pi},$$

$$\sqrt{F} = \frac{t_-^{(g)}}{2\pi} - a, \qquad c = F - a^2,$$

$$b = \frac{t_+^{(o)}\, t_-^{(o)}}{4\pi^2} - c.$$

The identity $\sqrt{F} = t_-^{(g)}/(2\pi) - a$ is positive in this class because $t_-^{(g)} > 2\pi a$. The winding scale κ is not inverted. The same typestate applies, with $c = (1 - \alpha^2)e^{2\beta x} < 0$, on the rotating family of Section whenever $\alpha > 1$. This algebraic reconstruction brings the present sheet into typestate parity with official Chrono-DM-A version 4 Theorem thm:jointID as a documentary comparison; that official identity is re-proved locally from Lemma and the printed host-null roots, and is not used as a theorem premise [1]. The statement is a declared kinematic map, not a data analysis and not a discharged observational test.

3. *(Ordinary fibre.) Ordinary guided-ring Sagnac data $(t_+^{(o)}, t_-^{(o)})$ alone still identify only a and $H_r = c + b$, and do not split c from b. A host-null datum, when supplied, is a named comparison on the host cone, not an ordinary-slot measurement. The theorem does not manufacture a host-bridge device, does not discharge an observational test as a data analysis, and does not identify a dark CTC from ordinary Sagnac times alone.*

Proof. The host-null equation at fixed (r, z) is $-\dot{t}^2 - 2a\dot{t}\dot{\varphi} + c\dot{\varphi}^2 = 0$, or equivalently $-(y + a)^2 + F = 0$. The inner product $g(y\partial_t + \partial_\varphi, \partial_t) = -y - a$ is negative precisely when $y > -a$, so only the root $-a + \sqrt{F}$ is future-directed for increasing φ. Integrating that root over $\Delta\varphi = +2\pi$, and the past root $-a - \sqrt{F}$ over $\Delta\varphi = -2\pi$, produces the formal times $2\pi(\sqrt{F} \mp a)$.

If $a > 0$ and $c > 0$, then $\sqrt{F} > a$, so both times are positive, which is . Their sum identifies $\sqrt{F}$, hence $c = F - a^2$; equivalently, their product is $4\pi^2(F - a^2) = 4\pi^2 c$. The identical product on $\hat{g}$ identifies H_r, and subtracting c identifies b. The difference of either pair is $4\pi a$.

If $a > 0$ and $c < 0$, then $\sqrt{F} < a$, so the formal co-rotating increment $2\pi(\sqrt{F} - a)$ is negative, while both roots are negative. A two-way pair of future arrivals is not obtained, and the product of those formal increments is not used as a future-arrival reconstruction. The counter-rotating future increment $t_-^{(g)} = 2\pi(\sqrt{F} + a)$ is nevertheless strictly positive. Lemma already reconstructs a and $H_r = t_+^{(o)} t_-^{(o)}/(4\pi^2)$ from ordinary data. Subtracting the ordinary reconstruction of a from $t_-^{(g)}/(2\pi)$ recovers $\sqrt{F}$, hence $c = F - a^2$, and $b = H_r - c$, which is –. On the rotating family with $\alpha > 1$, one has $c = (1 - \alpha^2)e^{2\beta x} < 0$ and $a = \alpha e^{\beta x} > 0$, which falls under this class.

Proposition (Host-null and ordinary ring times on $g_{\alpha,\beta}$). *On $(M, g_{\alpha,\beta})$ at a fixed (x, z), with $a = \alpha e^{\beta x}$ and $c = (1 - \alpha^2)e^{2\beta x}$, one has $F = e^{2\beta x}$ and $\sqrt{F} = e^{\beta x}$. The formal host-null increments are therefore*

$$t_+^{(g)} = 2\pi(1 - \alpha)e^{\beta x}, \qquad t_-^{(g)} = 2\pi(1 + \alpha)e^{\beta x}.$$

For every $\alpha > 1$ one has $t_+^{(g)} < 0$ and $t_-^{(g)} > 0$: only the counter-rotating host-null branch yields a future arrival after one revolution, which is the unidirectional signature of Theorem (b). If an ordinary weight satisfying Theorem is adjoined, the ordinary arrivals

$$t_\pm^{(o)} = 2\pi(\sqrt{\alpha^2 e^{2\beta x} + H(x)} \mp \alpha e^{\beta x})$$

are both strictly positive, and they identify a together with $H(x)$. Together with the single future host-null arrival $t_-^{(g)} > 0$, they reconstruct c and $b = H(x) - c$ by –. Ordinary-only data still do not split c from b. The product $t_+^{(g)} t_-^{(g)} = 4\pi^2 c$ remains an algebraic identity, and is not used as a future-arrival reconstruction in this class.

Proof. Direct substitution of $a = \alpha e^{\beta x}$ and $c = (1 - \alpha^2)e^{2\beta x}$ into $F = a^2 + c$ gives $F = e^{2\beta x}$. The formal increments of Theorem then reduce to , whose co-rotating member is negative precisely when $\alpha > 1$. The ordinary times are the same computation with c replaced by $H(x) > 0$, so $\sqrt{a^2 + H(x)} > a$ and both times are positive. Their difference is

$4\pi a$ and their product is $4\pi^2 H(x)$.

Proposition (Local vorticity comparison on the official exponential shield). *On $(M, g_{\alpha,\beta})$ with $\alpha > 0$ and $\beta \neq 0$, the unit Killing congruence $u = \partial_t = e_0$ is geodesic, and $\mathrm{d}\theta^0 = \alpha\beta\,\theta^1 \wedge \theta^2$. The vorticity scalar of this congruence, equivalently the Fermi–Walker precession rate of a spatial gyroscope relative to the co-rotating coframe, is the constant*

$$|\Omega|_g = \frac{1}{2}\alpha|\beta|.$$

On the official exponential ordinary shield of Theorem and Lemma , with $B(\mathrm{s}(x)) = B_0 e^{2\beta x}$ and $H(x) = (B_0\kappa^2 - \alpha^2 + 1)e^{2\beta x}$, the ordinary metric takes the form $\hat{g} = -(\mathrm{d}t + \alpha e^{\beta x}\,\mathrm{d}\varphi)^2 + \Delta_q e^{2\beta x}\,\mathrm{d}\varphi^2 + \mathrm{d}x^2 + \mathrm{d}z^2$ with constant $\Delta_q = 1 + B_0\kappa^2 > 1$. The ordinary coframe $\hat{\theta}^0 = \theta^0$, $\hat{\theta}^1 = \theta^1$, $\hat{\theta}^2 = \sqrt{\Delta_q}\,\theta^2$, $\hat{\theta}^3 = \theta^3$ then satisfies $\mathrm{d}\hat{\theta}^0 = \frac{\alpha\beta}{\sqrt{\Delta_q}}\hat{\theta}^1 \wedge \hat{\theta}^2$, so the ordinary vorticity scalar is the strictly smaller constant

$$|\Omega|_{\mathrm{o}} = \frac{\alpha|\beta|}{2\sqrt{\Delta_q}} = \frac{\alpha|\beta|}{2\sqrt{B_0\kappa^2 + 1}}.$$

The ratio

$$\frac{|\Omega|_{\mathrm{o}}}{|\Omega|_g} = \Delta_q^{-1/2}$$

therefore measures Δ_q by a local gyroscope comparison. The typestate of Theorem is not withdrawn: ordinary-only Sagnac data remain fibre-blind, κ is not inverted from two-way ring times alone, and the two-future-time product remains unused when $c < 0$. The comparison is not an observational five-channel acceptance set, and is not a replacement of official Chrono-DM-A Theorems thm:jointID and thm:jointIDwind [1], nor of Proposition .

Proof. The host identity $\mathrm{d}\theta^0 = \alpha\beta\,\theta^1 \wedge \theta^2$ is recorded in the proof of Theorem . For the geodesic unit Killing field $u = e_0$, the vorticity two-form on the rest space is $\frac{1}{2}\mathrm{d}\theta^0$, whose magnitude in the orthonormal coframe is $\frac{1}{2}\alpha|\beta|$. (Equivalently, writing $a(x) = \alpha e^{\beta x}$ and $F(x) = e^{2\beta x}$ as in Proposition , one has $|\partial_x a|/(2\sqrt{F}) = \frac{1}{2}\alpha|\beta|$, which is the same

scalar after the coframe computation.)

On the official exponential shield, $\widehat{g} = -\mathrm{d}t^2 - 2\alpha e^{\beta x}\,\mathrm{d}t\,\mathrm{d}\varphi + H(x)\,\mathrm{d}\varphi^2 + \mathrm{d}x^2 + \mathrm{d}z^2$ by the completing-the-square identity already used in Theorem , and $H(x) + \alpha^2 e^{2\beta x} = (1 + B_0\kappa^2)e^{2\beta x}$, which is the stated form. The ordinary observers ∂_t remain $\widehat{g}$-unit and $\widehat{g}$-geodesic, because $\widehat{g}_{tt} = -1$ is constant and the metric is stationary. Rescaling only the azimuthal coframe by the constant $\sqrt{\Delta_q}$ converts $\mathrm{d}\widehat{\theta}^0 = \alpha\beta\,\theta^1 \wedge \theta^2$ into $\frac{\alpha\beta}{\sqrt{\Delta_q}}\widehat{\theta}^1 \wedge \widehat{\theta}^2$, which is . The ratio follows at once. The Sagnac typestate is Theorem (b)–(c) together with Proposition .

Remark (Declared inverse problem). If both future host-null arrivals of Theorem (a) are supplied as data, together with the ordinary identification of *a*, the ordinary Sagnac fibre is reduced and the two printed completions with a fixed H_r are separated in the class $c > 0$. The model is to be rejected if $|(t_-^{(g)} - t_+^{(g)}) - (t_-^{(\mathrm{o})} - t_+^{(\mathrm{o})})|$ exceeds a declared consistency tolerance, or if the reconstructed *b* is significantly negative. In the class $c < 0$, the operational signature to be tested, if a host-null channel is independently supplied, is the absence of a co-rotating future arrival, not the sign of a two-time product. If that one-sided future arrival $t_-^{(g)}$ is independently supplied, the algebraic reconstruction – is the declared kinematic map, together with the consistency check $t_-^{(g)} > 2\pi a$.

Master synthesis and refusal boundaries

Theorem (Synthesis of official Dual-Ledger-Chrono-DM-B version 4). *Within the declared classical classes printed in Sections –:*

1. *the canonical switch kinetic action closes the four-slot conservation ledger* $\nabla_a(T_{\mathrm{o}}^{ab} + T_{\mathrm{d}}^{ab} + T_{\Theta}^{ab} + T_s^{ab}) = 0$ *as action bookkeeping, while the un-enlarged remainder is retained if that slot is omitted (Theorem);*

2. *the rotating family* $g_{\alpha,\beta}$ *has the constant orthonormal Einstein tensor , and admits three named residual-free Einstein-match branches, typestated by their sources, all with* $\Lambda = -\frac{1}{2}\beta^2 < 0$ *(Theorems –); canonical azimuthal winding stress is not cancelled (Proposition); a second closed one-form* $q_{(x)}$ *on the unchanged printed metric does not yield an on-shell residual-free branch (Proposition); official Branches G, L, and M are not withdrawn, and*

the Version 1 parameter set is withdrawn;

3. *for* $\alpha > 1$*, azimuthal circles are closed future-directed host-timelike curves with acceleration* βe_1*, hence not geodesics (Proposition), and the unshifted family admits no closed causal geodesic (Theorem), the official timelike typestate including being retained; an exponential (or faster) prescribed disformal weight yields a global ordinary Cauchy foliation, while a constant weight on a half-space does not (Theorem), and that exponential weight is realised by the two typestated on-shell test-switch profiles of Lemma ; comoving ordinary dust drops out of the switch equation, and a constant isotropic Jordan pressure is absorbed into a shifted potential, while a radially varying comoving pressure is not Jordan–Euler-on-shell (Proposition);*

4. *the bimetric Misner overlay, with a prescribed ramp, has a shared globally hyperbolic past, a compactly generated dark Cauchy horizon at* $T = 0$*, and ordinary global hyperbolicity on the region where* $H_M(T) > 0$ *(Theorem); an on-shell switch audit of* $s = s(T)$ *is recorded in Proposition , without withdrawing the prescribed typestate; future-directed causal geodesics split into a transmitting class* $J < 0$ *and an infinitely winding class* $J \geq 0$ *(Proposition); separated dark Klein–Gordon modes are analysed in Proposition , without unique dark continuation past* $T = 0$*;*

5. *joint two-future-time reconstruction of* (a, c, b) *is used only where both future host-null arrivals exist; in the class* $c < 0$*, ordinary two-way arrivals together with the single existing future host-null arrival* $t_-^{(g)} > 0$ *reconstruct* (a, c, b) *algebraically, without using the unphysical negative-*$t_+^{(g)}$ *product; the chronology-violating class remains typestated by unidirectional azimuthal cone tilt, and is realised explicitly on* $g_{\alpha,\beta}$ *for every* $\alpha > 1$ *(Theorem , Proposition); a local vorticity comparison on the official exponential shield measures* Δ_q *(Proposition), and does not withdraw fibre-blindness of ordinary-only Sagnac data. None of (i)–(v) manufactures a time machine from ordinary laboratory matter, produces backward signalling in the ordinary sector, derives a spacetime metric from quantum registers or access capacity, withdraws the official Chrono-DM-A flat or azimuthal-winding residuals, withdraws official residuals on* $g_{\alpha,\beta}$ *by a comparison written on a different metric family, or advances the inherited operational time-machine score from* 0 *of* 3*.*

Proof. Clauses (i)–(v) combine the theorems, propositions, and lemma cited in each item, together with Corollary as a restricted geometric comparison, Propositions , , , , , and as source-class, inventory, switch, and collar illustrations, and Propositions and Remark as the ordinary-sector continuation statement and the dark residual. Global hyperbolicity of the ordinary metrics in Theorems and , on the regions where the printed positivity conditions hold, precludes closed $\hat{g}$-causal curves for ordinary matter on those regions. The Standing Ledger refusals of Section remain intact. Track II programmes recorded in Appendix are not used as premises of (i)–(v).

appendix

Debugging official typestate and brief identities

The attached Dual-Ledger-Chrono-DM-B Version 1 is a source record, not a premise and not an official typestate. Draft scratch, mid-proof self-corrections, and several formally wrong identities occurred in that source. Official version 2 kept only complete local calculations, in official prose, and recorded the Version 1 corrections here. Official version 3 retains those Version 1 corrections, and records in addition the t507u brief identities that do not survive local checks, together with the Track II programmes that are not admitted into the spine. Official version 4 retains every official version 3 identity and refusal, including the official residual quarantine of Appendix , and records in addition the t509u brief identities that do not survive local checks, or that survive only after the honesty repairs printed in the spine.

Draft scratch removed from the official text

Version 1 contained mid-proof thinking and exclamation-heavy scratch, including phrasing of the form "Wait—we are on a quotient!", "Look at the geodesic equations", "100% identical", "Check sign below!", "Let us match … directly!", and unresolved sign questions inside the positive-split and Misner-shield arguments. None of that draft voice is retained in the theorems, remarks, or abstract of official version 2, official version 3, or official version 4. Failed or unfinished checks are recorded in this appendix rather than left in the official body.

Inconsistent parameter typestate

Version 1 did not print a single consistent $(\rho_{\mathrm{d}}, \Lambda, \text{Maxwell}, \text{winding})$ set.

1. The Version 1 abstract used $\rho_{\mathrm{d}} = M_{\mathrm{p}}^2\beta^2(4-\alpha^2)/2$ and $\Lambda = -M_{\mathrm{p}}^2\beta^2(2-\alpha^2)/2$, and claimed $T_{\mathrm{res}} \equiv 0$ with positive-norm winding stress. The cosmological label there carries a spurious factor M_{p}^2, and at $\alpha = \sqrt{2}$ that formula yields $\Lambda = 0$, which contradicts the Gödel dust match $\Lambda = -\frac{1}{2}\beta^2$. The winding-inclusive slogan is not realised by any admitted branch (Proposition).

2. Version 1 Theorem "positive-split"(a), at $\alpha = \sqrt{2}$, wrote a dust-only Einstein equation with $\Lambda = -\frac{1}{2}\beta^2$ and $\rho_{\mathrm{d}} = M_{\mathrm{p}}^2\beta^2$, then hedged with a "longitudinal magnetic harmonic two-form". The dust-only identity at $\alpha = \sqrt{2}$ is Theorem ; the hedge is not needed there.

3. The same Version 1 theorem first wrote $\rho_{\mathrm{d}} := \frac{1}{2}M_{\mathrm{p}}^2\beta^2(4-\alpha^2)$ and an undecided sign for Λ, then later adopted the set . Those three assignments are mutually inconsistent, and does not solve the Einstein equation for dust plus the printed Maxwell stress except at $\alpha = \sqrt{2}$, where the Maxwell amplitude vanishes (Theorem).

4. The Version 1 hand-check labelled "Som–Raychaudhuri limit $\alpha = 2$" claimed that at $\Lambda = 0$ matches the Einstein tensor with $\frac{1}{2}\mathscr{B}_0^2 = M_{\mathrm{p}}^2\beta^2$ and $\rho_{\mathrm{d}} = M_{\mathrm{p}}^2\beta^2$. Even internally, that hand-check disagrees with , which at $\alpha = 2$ gives $\rho_{\mathrm{d}} = 0$ and $\frac{1}{2}\mathscr{B}_0^2 = \frac{1}{2}M_{\mathrm{p}}^2\beta^2$. Neither assignment matches $G_{\hat{0}\hat{0}} = 2\beta^2$, $G_{\hat{1}\hat{1}} = \beta^2$, $G_{\hat{3}\hat{3}} = 0$ at $\Lambda = 0$. The correct Maxwell-plus-dust parameters at $\alpha = 2$ are those of , with $\Lambda = -\frac{1}{2}\beta^2 \neq 0$.

Table records the official assignments against the inconsistent Version 1 formulae. In every official branch, $\Lambda = -\frac{1}{2}\beta^2 < 0$. The Version 1 abstract, the first display in the Version 1 positive-split theorem, and disagree with one another and with the Einstein tensor of Theorem , except at the Gödel point where extra fields vanish.

Maxwell co-closure and the longitudinal alternative

Version 1 first claimed that $\mathscr{F} = \mathscr{B}_0\theta^1 \wedge \theta^2$ is co-closed, while writing an exterior derivative $\mathrm{d}\star_g\mathscr{F} \propto \alpha\beta\,\theta^1 \wedge \theta^2 \wedge \theta^3$ that does not vanish. The mid-proof correction to a longitudinal one-form $q_{(z)} = \kappa_z \mathrm{d}z$ is the

source class of Theorem , and is admitted for $\alpha \in (1, \sqrt{2}]$. The Maxwell field is admitted only as a sourced, force-free dark $U(1)_D$ current, in the class of Theorem . A source-free Maxwell supplement on the printed family is not obtained.

Closed curves versus closed geodesics

Version 1 stated that every azimuthal circle at $\alpha = 2$ is an exact closed future-directed host-timelike geodesic, and printed the acceleration formula $\nabla_v v = \frac{\beta(\alpha^2 - 2)}{\alpha^2 - 1} e_1$. That formula contradicts the Version 1 prose at the Gödel value (which claimed acceleration $-\beta e_1$ at $\alpha = \sqrt{2}$, whereas the printed formula vanishes there) and is false at every $\alpha > 1$. The coordinate computation, the Killing identity $\nabla_K K = -\frac{1}{2}\mathrm{grad}\,(K \cdot K)$, and the Christoffel component $\Gamma^x_{\varphi\varphi}$ all give $\nabla_v v = \beta e_1$ independently of α (Proposition). Circular geodesics are admitted only on the holonomy-shifted family, at one radius, for $0 < \alpha < 1$ (Corollary), a class that is not a positive-dust match in Theorem . Theorem further shows that the unshifted family itself admits no closed timelike geodesic at $\alpha > 1$.

Misner typestate

Version 1 left mid-proof sign questions in the time-orientation and ordinary temporal-function arguments, and styled the ordinary sector as globally hyperbolic "for all $T \in \mathbb{R}$ with zero Cauchy horizon". Official version 2 completes the sign checks (Definition and Theorem) and typestates ordinary global hyperbolicity as holding on the region where $H_M(T) > 0$. The compactly generated dark horizon at $T = 0$ remains. A prescribed ramp is not dynamical manufacture. Proposition is admitted as a mode analysis; the Version 1 suggestion of an intrinsic uniqueness rule is weakened to a regularity selection, and the unique-continuation residual is kept. Proposition records the corresponding classical geodesic and Vlasov dichotomy on the same prescribed overlay, and is not a dynamical back-reaction.

Sagnac product identity

Version 1 used the product $c = t_+^{(g)} t_-^{(g)}/(4\pi^2)$ as a reconstruction in all sign classes of c, including $c < 0$. Official Chrono-DM-A already typestated that both future host-null arrivals need not exist when $c < 0$, and that the

fibre-breaking signature is then unidirectional cone tilt [1]. Official version 2 follows that typestate (Theorem) and does not treat the product as a future-arrival reconstruction outside the class $c > 0$. Official version 3 keeps that refusal, and upgrades the class $c < 0$ by the algebraic reconstruction – from ordinary two-way data together with the single existing future arrival $t_{-}^{(g)} > 0$.

"Obstructions resolved" language

Version 1 titled its principal theorems as resolutions of "obstructions L1–L4" and summarised them as residual-free positive-source splits, horizon-free ordinary evolution, and complete Sagnac identifiability. Official typestate does not treat those official A residuals as completely discharged: the switch ledger was already closed as bookkeeping; the flat and azimuthal-winding residuals stand; the Misner overlay is prescribed, not manufactured; and Sagnac identifiability is class-limited. The Version 1 master theorem and summary table are therefore rewritten as Theorem and the table in Section .

t507u brief identities found formally wrong

The t507u brief proposed four Track I enhancements and four Track II breakthroughs. Official typestate follows the locked t507u plan, not the brief's discharge language. The following brief formulae fail a local check, or must be typestated more narrowly than the brief asserted, and are not pasted into the spine.

1. The brief period $T_x = 2\pi/(\beta\sqrt{E^2 - P_z^2 - 1})$ of a radial bounce on $g_{\alpha,\beta}$ is formally wrong. The official period, derived from the first integrals of Theorem (a), is . The two expressions are not proportional independently of α and E.

2. The brief claim that $\Delta t > 0$ can be read off from that incorrect period is not used as a proof. The official increment is derived independently, and is then shown to be strictly positive for every future-directed bounce with $\alpha > 1$ and $E > 0$ by the comparison $\sqrt{\alpha^2 - 1} < \alpha$.

3. The linear on-shell switch of the brief, with $W(s) = F_s^2 \beta s_0 s + W_0$, solves , but W is unbounded below on $\mathbb{R}$. Official Lemma (i) records that unboundedness and does not call the profile a non-negative potential. The brief's

display $s(x) = s_\infty + s_1|\beta|^{-1}e^{-\beta x}$ is a sign-special case of the official integral ; the official typestate uses the latter, and records the divergence of $B(s)$ as $s \to s_\infty$ at one end of the x-line.

4. On the prescribed Misner overlay, the brief slogans that the class $J \geq 0$ has $\Delta\chi \to -\infty$ and never reaches $T = 0$, with a uniform $1/|T|$ energy divergence, do not survive the official future branch. Proposition typestates $\Delta\chi \to +\infty$, finite proper time with infinite winding and no C^1 extension through $\mathscr{H}_0$, and an energy divergence of order $1/|T|$ only for $J > 0$ (of order $|T|^{-1/2}$ for $J = 0$). The crossing formula $\dot{\chi}(0) = -(1 + P_\perp^2)/(2J)$ is admitted only after reduction of the official mass shell, and is checked against the official $J = -1$ root.

5. Track II brief identities are recorded in the next subsection; in particular, the mid-brief Maxwell closure for an anisotropic two-form is retracted by the brief itself, and the pasted Proca stress $\frac{1}{2}m_V^2\mathscr{A}_0^2\mathrm{diag}\,(+1,-1,-1,-1)$ is not admitted.

t509u brief identities found formally wrong or refused

The t509u brief proposed five Track I enhancements on the printed family, and four Track II companion packages. No official version 3 identity was found formally wrong. The following brief identities do not survive local checks as stated, or are typestated only after the honesty repairs already printed in the spine.

1. The brief null period $T_{x,\mathrm{null}} = 2\pi/(|\beta|E\sqrt{\alpha^2 - 1})$ and the increment $\Delta t_{\mathrm{null}} = \frac{2\pi}{|\beta|}(\alpha - \sqrt{\alpha^2 - 1})$ are the formal $\varepsilon = 0$ analogues of the official timelike identities, and they are admitted in Theorem only after recomputation from official Theorem (a), including the cases $E \leq 0$, constant-x null candidates, and $P_z \neq 0$. They were not pasted before that check. The slogan "causal geodesic completeness of the obstruction" is refused: geodesic completeness of the manifold is already Theorem (a), and closed non-geodesic causal curves already exist by Proposition .

2. The brief mid-argument "wait" in the pressure calculation is scratch, not a proof, and is not retained. The prefactor survives after derivation from official $\hat{g}^{ab}$ and $\mathscr{T}^{ab} = \sqrt{\Delta_q}\,\hat{T}_o^{ab}$. The claim that a radially decaying comoving pressure realises the global ordinary Cauchy condition together with four-slot conservation is refused: such a pressure is not Jordan–Euler-on-shell for the geodesic congruence ∂_t (Proposition).

3. The brief stress sum $\mu^2 \mathrm{diag}\,(+1, 0, 0, -1)$ is not the official Maxwell stress, and is not called a Maxwell signature. The brief dust formula containing a mid-argument check mark, the assignment $\mu^2 = \frac{1}{4} M_{\mathrm{P}}^2 \beta^2 (\alpha^2 - 2)$, and the official Branch M value $\Lambda = -\frac{1}{2}\beta^2$ for this two-form pair, are not pasted. The stiffness profile $M_x(x) = M_{x,0} e^{-\beta x/2}$ does make $\nabla_a (M_x^2 q^a_{(x)}) = 0$, but then μ_x^2 is nonconstant, so a residual-free match on all of M fails (Proposition).

4. The brief identities $\mathrm{Box}_{g_M} s = Ts'' + s'$ and $(\nabla s)^2 = T(s')^2$ survive after derivation from the official inverse metric, and are admitted in Proposition . The sign change of $(\nabla s)^2$ across $T = 0$ is typestated, and the linear potential is typestated as unbounded below. A "scratch-free" residual-free horizon crossing is not claimed, and the audit is not Track II Breakthrough 4.

5. The brief gyroscope formulae $|\Omega|_g = \frac{1}{2}\alpha|\beta|$ and $|\Omega|_o / |\Omega|_g = \Delta_q^{-1/2}$ survive after derivation from the official coframes, and are admitted in Proposition . They were not pasted as $|\Omega| = |\partial_x a|/(2\sqrt{F})$ before that check. Ordinary-only Sagnac fibre-blindness is not withdrawn.

Track II programmes, recorded only as named comparisons or refusals

The four Track II programmes of the t507u brief, restated in the t509u brief as proposed standalone companions, target official residuals on the printed family, or on neighbouring constructions. None is folded into the B-spine as a residual-resolving flagship. None withdraws Proposition , Proposition , Proposition , Theorems , , or , or the typestate $\Lambda = -\frac{1}{2}\beta^2$ on the official branches of $g_{\alpha,\beta}$. A match written on a different metric family does not cancel a residual computed on $g_{\alpha,\beta}$. None is Bar 3, and no companion package is created from this brief: in particular, no Chrono-Bianchi-Winding-A, Chrono-Holonomy-Proca-A, Chrono-SomRaychaudhuri-A, or Chrono-Misner-Dyn-A sheet is written.

1. *(Anisotropic homogeneous enlargement $g_{\alpha,\beta,\gamma}$.)* The brief proposes the prescribed ansatz

$$g_{\alpha,\beta,\gamma} := -(\mathrm{d}t + \alpha e^{\beta x}\,\mathrm{d}\varphi)^2 + \mathrm{d}x^2 + e^{2\beta x}\,\mathrm{d}\varphi^2 + e^{2\gamma x}\,\mathrm{d}z^2 \qquad (\gamma \neq 0)$$

as a winding-inclusive exact Einstein match, after retracting an incorrect Maxwell closure mid-argument. This is a

different homogeneous family from official $g_{\alpha,\beta}$. It does not withdraw Proposition on the printed family. No complete local Einstein-tensor computation of $g_{\alpha,\beta,\gamma}$, and no residual-free source match, is admitted in this sheet. The programme is recorded as a named comparison on a different prescribed ansatz, not as a resolution of official Proposition 4.3, not as a strictly positive source, not as Einstein dynamics, and not as Host C. No official theorem titled as a residual-free winding match on this foreign metric is admitted, and no Chrono-Bianchi-Winding-A companion is created.

2. *(Genuine Som–Raychaudhuri comparison.)* Official typestate already treats the Som–Raychaudhuri cylinder as a named comparison and refuses $\Lambda = 0$ on the homogeneous family $g_{\alpha,\beta}$ (Proposition and Theorem). The literature Einstein–Maxwell plus charged rigid dust solution [6] may be cited as a source record in that sense. An axis-regular $\mathbb{R}^4$ vortex time machine is not pasted, official Chrono-DM-C is not rewritten, unwritten official Chrono-DM-C version 3 is not cited, and no Dark-Vortex-CTC-A or Chrono-SomRaychaudhuri-A package is created. $\Lambda = 0$ is not obtained on $g_{\alpha,\beta}$. The comparison is not Bar 3.

3. *(Proca or two-fluid match on the holonomy-shifted family at* $0 < \alpha < 1$*.)* Corollary already admits closed circular geodesics on g_{α,β,A_0} for $0 < \alpha < 1$, and official Branch L requires $\alpha > 1$ for positive dust on the unshifted family. A new source class on the different metric $A_0 \neq 0$ does not withdraw official Branches G, L, and M. The brief's pasted Proca stress $\frac{1}{2}m_V^2\mathscr{A}_0^2\mathrm{diag}\,(+1,-1,-1,-1)$, for $\mathscr{A} = \mathscr{A}_0\theta^0$, is not admitted: already $\mathrm{d}\mathscr{A} = \mathscr{A}_0\,\mathrm{d}\theta^0 = \mathscr{A}_0\alpha\beta\,\theta^1\wedge\theta^2$ is nonzero, so a mass-term-only stress is incomplete, and the mass-term contribution itself is not of that diagonal form in the official orthonormal frame. The t509u brief's mid-argument index-raising "check" is not a proof, and is not retained. Even a complete source-free Proca identity at $m_V = |\alpha\beta|$, were it established, would live on the holonomy-shifted family g_{α,β,A_0} and would not withdraw official residuals on unshifted $g_{\alpha,\beta}$, official Branches G, L, and M, or the typestate of Corollary . Closed geodesics at $0 < \alpha < 1$ on the holonomy-shifted family remain that corollary's class, not Bar 3. No WEC or NEC claim is pasted for this unfinished class, especially while Λ remains negative on the homogeneous curvature. The programme is recorded as a refusal of the pasted stress, not as a simultaneous positive-dust and closed-geodesic match on the official

branches, and no Chrono-Holonomy-Proca-A companion is created. It is not Bar 3.

4. *(Back-reacting Kasner/Gowdy-to-Misner programme.)* The brief proposes an exact coupled Einstein–switch solution g_{dyn} with closed quadratures and zero residual across a dynamically formed horizon. Official Theorem remains a prescribed overlay with a prescribed ramp. No exact coupled Einstein solution, and no dynamically formed horizon with zero residual, is pasted. The slogan $T_{\mathrm{res},\,ab} \equiv 0$ and a "scratch-free" horizon crossing are not admitted. Proposition is not this breakthrough. The programme is recorded as unfinished, and no Chrono-Misner-Dyn-A package is created. It is not Bar 3.

Bibliographic and verification discipline

This official version 5 is a typographic successor of official version 4 (request t509u). Official version 4 typestate, labels, and local proofs are retained. Authorised Version 5 work is typestate hygiene, ordinary quotation marks, a Scholar-parseable bibliography without an invented DOI, and the documentary negative-control table of Section . Siblings remain source records at their official currently delivered versions; unwritten t563u successors are not cited. Card 11 remains 0 of 3.

Official Dual-Ledger-Chrono-DM-A version 4 (request t506u), official Dual-Ledger-Mobility-A version 5 (request t495u), and official Dual-Ledger-Temporal-A version 5 (request t504u) are documentary source records, not previously compiled premises and not theorem imports [1, 2, 3]. Official Chrono-DM-C version 3 is not cited. The typographic parent described that neighbour as unwritten; official version 3 has since been delivered, and the present sheet still does not cite it. Confirmed published DOIs already present in the attached Version 1 bibliography (Bekenstein, Gödel, Som–Raychaudhuri, Reboucas–Tiomno) are retained after checking titles [4, 5, 6, 7]. No DOI, figshare, Zenodo, or Scholar page is invented. No companion landing-page URL is invented, and no deposit or Scholar page is asserted. No compilation or machine verification is asserted. Informal environment and brace checks, if recorded, are not a TeX compile.

Declarations

Authorship and submission. The author of this official sheet is Ngo Cheung. No affiliation is invented. No endorsement by a cited author or publisher is inferred. Book, figshare, arXiv, and Zenodo deposit, and any Scholar campaign, remain out of scope.

Physical assumptions. The host metrics, quotient identifications, winding sector, actions, and matter closures are declared. Einstein source compatibility, when considered, is an independently stipulated equation on a prescribed ansatz. It is host-compatibility bookkeeping, not Einstein dynamics, and not Host C. No dark-matter species or quantum-gravitational mechanism is selected. wired=false. declaredPath remains null.

Proof scope. All principal geometric and action assertions are proved under their printed hypotheses, including Theorems , , , , , , , and , and Propositions , , , , , and , and Lemmas and . The source-compatible manufacture of a dark CTC from laboratory initial data, unique field well-posedness on a chronology-violating host, a continuum derivation of either finite CTC rule, a localised vortex-ring host on $\mathbb{R}^4$, and physical clock acquisition are not claimed. The Gödel-type match is not a winding-inclusive exact match, and the Misner overlay is not dynamical manufacture. Ordinary Sagnac data alone do not identify a dark CTC.

Standing ledger. Host C is not created. O.1 stays NOT_CLOSED. XV-B is not opened. XVI-N and Route B are not discharged except the class-limited negative of t441 XVI-N-Univ-A. Gate 0 and ST4b stay NOT_SUPPLIED. H3, A1–A3, and the phenomenal-theory gate remain NOT_CLOSED and unused. No quantum coefficient, derived metric, spacetime emergence, Page curve, island, bounce, or vacuum-energy claim is made. The inherited time-machine bar remains 0 of 3 as the full three-part bar. A prescribed Einstein-compatible TM-2 witness, including a Gödel-type or Som–Raychaudhuri collar, a prescribed Misner ramp, an on-shell test-switch profile, a closed causal-geodesic obstruction, a Misner switch-stress audit, a local gyroscope comparison, and any Track II appendix comparison, is not Bar 3 and does not advance the score. Companion records and the OpenAI mathematics catalogue (supplied documentary record dated 7 October 2026) are source records only, not premises.

Documentary negative-control catalogue

The following one-page table isolates unverified external catalogue entries and refused slogans as documentary negative controls only [10]. They are not theorem premises, not scheduler-derived cones, not a time machine, not a Page curve, not a free clock, not worldline return, and not a decoder of an unread complement. They do not move the ordinary operational time-machine bar, which remains 0 of 3, and they do not typestate uniqueness of dark matter or a chronology engine in nature. Clock Track-I identities are not imported from this catalogue, and no Host C, quantum coefficient, island, bounce, or de Sitter state is typestated.

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[1] Cheung, N. (2026). *Dual-Ledger-Chrono-DM-A: Winding, Chronology Asymmetry, and Operational Insulation on Prescribed Lorentzian Hosts. Exact Geometric Witnesses, Corrected No-Paradox Results, and Complete-Record Rejection Contracts*. Enter Calculus Companion, official version 4 (request t506u). *Source record; not a premise.*

[2] Cheung, N. (2026). *Dual-Ledger-Mobility-A: Mobility asymmetry between an ordinary slot and a current-free dark slot on a prescribed host: a conditional fluid-plus-controller class, its conservation ledger, and declared inverse problems*. Enter Calculus Companion, official version 5 (request t495u). *Source record; not a premise.*

[3] Cheung, N. (2026). *Dual-Ledger-Temporal-A: Sector-Dependent Clock Rates and Transport on a Prescribed Lorentzian Host. Conservation, No-Freezing Results, Controlled Cosmological Witnesses, and Observational Rejection Contracts*. Enter Calculus Companion, official version 5 (request t504u). *Source record; not a premise.*

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[10] OpenAI mathematics catalogue entries Oai365-A (Family 365), Oai290-A (Family 290), Oai265-A/B, Oai276-A, Oai212-A, and refused Family 260. Documentary negative controls only, isolated to Section ; never premises. Family 260 remains refused.