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Dual-Ledger-Chrono-DM-C

Continuum Dark-Field Self-Consistency, Detector Seam Rules, and a Prescribed Toroidal Chronology Split on $\mathbb{R}^4$

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Abstract

Dual-Ledger-Chrono-DM-A established bimetric chronology splits on cylindrical quotients with global S^1 topology, and treated finite Deutsch and postselected closed-timelike-curve maps as separately postulated toy rules. Official version 3 is a typestate-honest officialisation of official version 2. The present companion sheet is official version 4, a typographic successor of official version 3 [1]. Official version 3 typestate, labels, and local proofs are retained. The typographic parent already repairs a coordinate-inconsistent single-invariant on-shell pullback, and records typestate-honest local enhancements in declared analytic classes, without discharging the unfinished programmes of Appendix . Official Version 4 does not upgrade those identities to a manufactured-core or operational-CTC theorem. None of these results manufactures a time machine, derives Host C, or advances the inherited operational bar.

Results. (1) *Continuum dark-field self-consistency on a prescribed CTC quotient.* On the chronology-violating host (M, g) whose constant-t hypersurfaces fail to be spacelike ($g^{tt} > 0$), t-Cauchy evolution for a dark scalar is not a hyperbolic initial-value problem. We prove that $\mathrm{Box}_g - m_{\mathrm{d}}^2$ is strictly hyperbolic with respect to any covector in the g-timelike dual cone, including the globally defined closed covector $\xi_0 = \mathrm{d}t + a\,\mathrm{d}\varphi$, while no globally single-valued smooth time function exists on the quotient (Theorem). On compactified time-periodic blocks, a Banach contraction in H^s ($s > 2$) yields a unique smooth self-consistent periodic solution in a uniform spectral-gap class, which is inhabited by an explicit parameter family (Theorem and Remark). A measure-generic Diophantine

bound with derivative loss is recorded for the linear resolvent; the nonlinear contraction is not extended by a Nash–Moser argument (Proposition). On the true non-compact cylinder, a limiting-absorption and Yukawa resolvent analysis is given for the prescribed stationary operator (Proposition). (2) *Microscopic Unruh–DeWitt seam bridge.* For a *d*-level internal dark detector transported along a closed future-directed *g*-timelike geodesic and coupled on a localized patch to an ordinary register *S* on $(M, \widetilde{g})$, kinematic reduced-state periodicity yields Deutsch's fixed-point map, whereas coherent Hilbert-space trace periodicity yields the postselected Kraus operator $A_{\mathrm{loop}} = \frac{1}{d}\mathrm{Tr}_D\, U$ (Theorem). Replacing the joint unitary by a printed bilinear Unruh–DeWitt interaction, the first-order Magnus operator is computed, and a closed analytic zero-locus for the postselected denominator is obtained in a commuting involutory-dipole class (Theorem (a)–(b)). In the same traceless involutory class, Deutsch's map admits a continuous family of μ_D-commuting fixed points and dephases ρ_S at the maximally mixed dark state, whereas the postselected evaluation is the identity channel whenever $\cos\theta \neq 0$; a second-order Magnus term is computed in a printed non-commuting dark Floquet class (Theorem (c)–(d)). These remain finite-detector seam comparisons, not continuum derivations of either finite rule, and not chronology theorems. (3) *Prescribed bimetric toroidal chronology-split witness on* $\mathbb{R}^4$. On topologically trivial $M = \mathbb{R}^4$ we prescribe a smooth, compactly supported unit-lapse frame-dragging host, equal to Minkowski space outside a bounded spatial ball, containing an interior solid-torus collar of closed future-directed *g*-timelike curves and at least one closed *g* -timelike geodesic. A globally smooth complex controller $\Phi = h(r^2, z^2)(x + \mathrm{i}y)$ removes the axis singularity of $\mathrm{d}\varphi$. Its Euler–Lagrange operator is computed locally, and an on-shell potential-and-switch realisation is exhibited by a two-invariant potential on (r^2, z^2) under an integrability condition, or unconditionally on a *z*-independent radial collar (Theorems and). The official version 2 single-invariant pullback to $\sigma = (r - r_0)^2 + z^2$ is withdrawn as not coordinate-consistent. The ordinary metric $\widetilde{g} = g + B_0 J[\Phi] \otimes J[\Phi]$ is globally hyperbolic on $\mathbb{R}^4$ and coincides with Minkowski space outside the same ball, once the disformal coupling exceeds a corrected quartic threshold (Theorem). Linearised transverse Jacobi equations about the closed ring are computed; a strict local maximum of $C_- = -g_{\varphi\varphi}$ does not by itself yield a linearly stable epicyclic tube (Proposition). A printed cratered profile, in

which C_{-} has a positive local minimum inside the CTC tube, is exhibited for which both official Jacobi coefficients are positive (Proposition). (4) *Unit-lapse Einstein-tensor typestate, stationary solitons, and annular delays.* The orthonormal Einstein tensor of the printed unit-lapse ansatz is recomputed locally. The energy density is $G_{\hat{0}\hat{0}} = +\frac{3}{4}|\mathbf{B}_{\mathrm{gm}}|^2 \geq 0$, correcting a Version 1 statement that printed the opposite sign, while the null energy condition can still fail in a compact-support curl shell (Proposition). A linear convected Helmholtz / magnetic Schrödinger reduction of dark Klein–Gordon on the same ansatz is recorded as an operator identity, not as nonlinear existence, Einstein matching, or unique continuation across a chronology horizon (Proposition). At vanishing frequency the magnetic potential drops and a small-data, or small-coupling, Yukawa contraction yields a unique stationary cubic solution in a printed Sobolev class (Proposition); this is not a discharge of time-periodic nonlinear existence on $(\mathbb{R}^4, g)$. First-order equatorial arrival delays for a compact annular profile are computed as Abel integrals of the drag and of the disformal amplitude, with an odd/even separation of $(w, B_0 h^4)$ (Proposition); ordinary-only data remain fibre-blind in the sense of the official Sagnac typestate, and exact finite-amplitude C^∞ deflection angles are not claimed.

Standing ledger. Host A and Host B stay separate prescribed backgrounds. Host C is not created. O.1 stays NOT_CLOSED. XV-B is not opened. XVI-N and Route B are not discharged, except the class-limited negative of t441 XVI-N-Univ-A. Gate 0 and ST4b stay NOT_SUPPLIED. No quantum coefficient is admitted as a chronology theorem. No derived metric, spacetime emergence, Page curve, island, bounce, or vacuum-energy claim is made. The inherited operational time-machine bar remains 0 of 3 as the full three-part bar. A prescribed compactly supported toroidal TM-2 witness on $\mathbb{R}^4$, even if the ordinary sector is globally hyperbolic, is not Bar 3 and does not advance the score. H3, A1–A3, and the phenomenal-theory gate remain NOT_CLOSED and unused. Companion records and the OpenAI mathematics catalogue (supplied documentary record dated 7 October 2026) are source records only, not premises. wired=false; declaredPath remains null. No compilation or machine verification is asserted.

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Scope and relation to official Chrono-DM-A

In official Dual-Ledger-Chrono-DM-A version 4 [2] [Source record; not a premise.], bimetric chronology asymmetry was realized using closed one-forms $q = \kappa\, \mathrm{d}\varphi$ on cylindrical manifolds $M = \mathbb{R}_t \times \mathbb{R}_x \times S^1_\varphi \times \mathbb{R}_z$, while quantum chronology insulation was analysed via separately declared Deutsch [5] and postselected [6] finite-register maps. Official typestate from that sheet remains in force as a documentary neighbour: Propositions "flat residual" and "Gödel residual" are not withdrawn here; the host-null guided-ring reconstruction applies only where both future host-null arrivals exist; a prescribed Misner or toroidal overlay is not dynamical manufacture; and Deutsch and postselected maps are toy rules, not continuum derivations, unless a complete new proof honestly says otherwise. Official Dual-Ledger-Chrono-DM-B version 4 [3] [Source record; not a premise.] is a documentary neighbour of a separate commission. It is not typestated here as a winding-inclusive residual-free Einstein match, and no identity is imported from that sheet as a premise. Official Dual-Ledger-Mobility-A version 5 [4] [Source record; not a premise.] is likewise a documentary source record only; no mobility identity is used as a premise.

Two structural questions remained at the end of those source records.

1. *Continuum dark-field solvability and the continuum-to-register gap.* On a host (M, g) containing closed timelike curves, constant-t slices are not Cauchy surfaces ($g^{tt} > 0$ in a CTC collar), so standard t-initial-value problems for continuum dark fields $(\mathrm{Box}_g - m_{\mathrm{d}}^2)\psi_{\mathrm{d}} = F(\psi_{\mathrm{d}})$ are not hyperbolic in t. Under what precise analytic conditions do smooth self-consistent continuum dark fields exist, and how do Deutsch's density-matrix fixed point and the postselected Kraus contraction arise as seam conditions for a localized worldline coupling on $(M, g, \hat{g})$?

2. *Topological triviality and spatial localization on* $\mathbb{R}^4$. By the closed-orbit Rolle obstruction proved in official Dual-Ledger-Chrono-DM-A [2] [Source record; not a premise.], no globally single-valued real scalar $\varphi \in C^\infty(\mathbb{R}^4; \mathbb{R})$ can uniformly narrow the light cones along a closed g-timelike loop via $\mathrm{d}\varphi \otimes \mathrm{d}\varphi$. On $\mathbb{R}^4$, the naive azimuthal one-form $\kappa\,\mathrm{d}\varphi$ is singular along the axis $\{x = y = 0\}$. Can a completely smooth, singularity-free field theory on topologically trivial $M = \mathbb{R}^4$ confine closed dark timelike geodesics inside a bounded spatial torus while keeping $(\mathbb{R}^4, \hat{g})$ everywhere smooth, globally hyperbolic, and identically Minkowskian outside a finite ball?

Sections and address inside declared analytic and finite-detector classes. Section addresses by a prescribed bimetric witness, not by laboratory manufacture. Every operative statement is proved locally from definitions printed in this manuscript. Classical self-consistency analyses on CTC manifolds [7, 8] and asymptotically flat toroidal CTC geometries [9] are cited as established background literature, not as transferred proofs.

Version 1 of the present sheet contained typographic and typestate defects which official version 2 repairs on the record. The macros backslash Tres and backslash bm (and, for typographic self-containment, also backslash norm, backslash Ttot, and backslash supp) are defined in the preamble. Conversational scratch, retracted metrics, and mid-proof sign reversals have been removed from official statements. The orthonormal Einstein tensor of the unit-lapse ansatz has been recomputed locally: the Version 1 proposition statement printed $G_{\hat{0}\hat{0}} = -\frac{1}{4}|\mathbf{B}_{\mathrm{gm}}|^2 \le 0$, which is false for the printed coframe metric; the correct density is $+\frac{3}{4}|\mathbf{B}_{\mathrm{gm}}|^2 \ge 0$, and the null energy condition is typestated separately. A Version 1 threshold $B_0 > \Omega_0^2 r_{\mathrm{out}}^2$ is insufficient for $\hat{g}_{\varphi\varphi} > 0$; the corrected sufficient bound is

$B_0 > \Omega_0^2 r_{\text{out}}^4$. A Version 1 evaluation of $\mathrm{Tr}_D U$ for the controlled-Hadamard SWAP portal is corrected. Failed or unfinished structural programmes are recorded in Appendix , not in official theorem statements.

Official version 3 keeps every official version 2 correction just recorded, and repairs one official version 2 identity that does not survive a local check. Official Version 4 retains that official version 3 typestate. Theorem of official version 2 pulled $\lambda(r^2, z^2)$, $|\Phi|^2 = r^2 h^2$, and $\Delta_{\mathbb{R}^3} s$ back to functions of a single toroidal invariant $\sigma = (r - r_0)^2 + z^2$, and claimed that $W'(s)$ was a function of s alone. That pullback is not coordinate-consistent on $\mathbb{R}^3$: r^2 and the cylindrical $1/r$ derivatives vary along level tori of σ, and a compactly supported switch is typically non-monotone. Official version 3 replaces the claim by a two-invariant potential on (r^2, z^2) under an integrability condition, or by a z-independent radial collar on which the remaining invariant is actually a coordinate for the potential. Track I enhancements—a printed cratered profile with linearly oscillatory official Jacobi coefficients, a finite-detector Deutsch versus postselected contrast together with a second-order Floquet Magnus term, a small-data stationary cubic contraction, and first-order annular Abel delays—enter the spine only after local checks, at the typestate actually obtained. Official residuals of version 2, including Proposition , are not withdrawn. Failed or unfinished structural programmes, including Breakthroughs 1–4 of the commissioning brief, remain in Appendix and are not official theorems.

Continuum dark-field self-consistency on the CTC quotient

Recall from official Dual-Ledger-Chrono-DM-A [2] [Source record; not a premise.] the complete cylindrical host $M = \mathbb{R}_t \times \mathbb{R}_x \times S^1_\varphi \times \mathbb{R}_z$ ($\varphi \sim \varphi + 2\pi$) with constants $a > 0$ and $0 < h < a^2$:

$$g = -\mathrm{d}t^2 - 2a\,\mathrm{d}t\,\mathrm{d}\varphi - h\,\mathrm{d}\varphi^2 + \mathrm{d}x^2 + \mathrm{d}z^2.$$

Put $\Delta_0 := a^2 - h > 0$. Direct inversion of the (t, φ) metric block $\begin{pmatrix} -1 & -a \\ -a & -h \end{pmatrix}$ (determinant $-\Delta_0 < 0$) gives the exact contravariant metric

$$g^{tt} = \frac{h}{\Delta_0} > 0, \qquad g^{t\varphi} = -\frac{a}{\Delta_0}, \qquad g^{\varphi\varphi} = \frac{1}{\Delta_0} > 0, \qquad g^{xx} = g^{zz} = 1.$$

Because $g^{tt} = h/\Delta_0 > 0$, the level sets $\{t = t_0\}$ are timelike hypersurfaces of (M, g), and the wave operator

$$\mathrm{Box}_g \psi_{\mathrm{d}} = \frac{h}{\Delta_0}\partial_t^2\psi_{\mathrm{d}} - \frac{2a}{\Delta_0}\partial_t\partial_\varphi\psi_{\mathrm{d}} + \frac{1}{\Delta_0}\partial_\varphi^2\psi_{\mathrm{d}} + \partial_x^2\psi_{\mathrm{d}} + \partial_z^2\psi_{\mathrm{d}}$$

is not hyperbolic with respect to t.

Helical principal-symbol hyperbolicity versus the quotient obstruction

Theorem (Dual-cone hyperbolicity and global periodic obstruction on (M, g)). *On (M, g) with –:*

1. *(Exact dual timelike cone.) A covector $\xi = \xi_t \mathrm{d}t + \xi_\varphi \mathrm{d}\varphi + \xi_x \mathrm{d}x + \xi_z \mathrm{d}z$ is g-timelike ($g^{ab}\xi_a\xi_b < 0$) if and only if*

$$\frac{(\xi_\varphi - a\xi_t)^2}{\Delta_0} - \xi_t^2 + \xi_x^2 + \xi_z^2 < 0.$$

In particular, for $\xi_x = \xi_z = 0$, the covector ξ is g-timelike if and only if $|\xi_\varphi - a\xi_t| < \sqrt{\Delta_0}\,|\xi_t|$. The closed covector $\xi_0 := \mathrm{d}t + a\,\mathrm{d}\varphi$ satisfies $g^{-1}(\xi_0, \xi_0) = -1 < 0$. On the universal cover $\widetilde{M} = \mathbb{R}^4$, the tilted coordinate $\tau_{\mathrm{hel}} := t + a\varphi$ is a global Cauchy time function, and the wave operator takes the standard Lorentzian form

$$\mathrm{Box}_g = -\partial_{\tau_{\mathrm{hel}}}^2 + \frac{1}{\Delta_0}\partial_{\widetilde{\varphi}}^2 + \partial_x^2 + \partial_z^2.$$

2. *(Quotient obstruction to helical Cauchy slices.) On the quotient $M = \mathbb{R}_t \times \mathbb{R}_x \times S^1_\varphi \times \mathbb{R}_z$, the function $\tau_{\mathrm{hel}} = t + a\varphi$ is multi-valued ($\tau_{\mathrm{hel}} \mapsto \tau_{\mathrm{hel}} + 2\pi a$ around S^1_φ). Every globally single-valued smooth function $F : M \to \mathbb{R}$ satisfies $\oint_{S^1_\varphi} \partial_\varphi F\,\mathrm{d}\varphi = 0$, so at any point where $\partial_\varphi F = 0$ one has $g^{-1}(\mathrm{d}F, \mathrm{d}F) = \frac{h}{\Delta_0}(\partial_t F)^2 + (\partial_x F)^2 + (\partial_z F)^2 \geq 0$. Consequently (M, g) admits no globally single-valued smooth time function.*

Proof. (a) Expanding gives $g^{ab}\xi_a\xi_b = \frac{1}{\Delta_0}(h\xi_t^2 - 2a\xi_t\xi_\varphi + \xi_\varphi^2) + \xi_x^2 + \xi_z^2$. Writing $h = a^2 - \Delta_0$ turns the numerator into $(a^2 - \Delta_0)\xi_t^2 - 2a\xi_t\xi_\varphi + \xi_\varphi^2 = (\xi_\varphi - a\xi_t)^2 - \Delta_0\xi_t^2$, which is . Under $\tau_{\mathrm{hel}} = t + a\varphi$ and $\widetilde{\varphi} = \varphi$, one has $\partial_t = \partial_{\tau_{\mathrm{hel}}}$ and $\partial_\varphi = a\partial_{\tau_{\mathrm{hel}}} + \partial_{\widetilde{\varphi}}$, so $(\xi_\varphi - a\xi_t)^2/\Delta_0 - \xi_t^2$ maps at the symbol level to $\Delta_0^{-1}\partial_{\widetilde{\varphi}}^2 - \partial_{\tau_{\mathrm{hel}}}^2$, which is .

(b) Along each closed circle $\gamma(\varphi) = (t_0, x_0, \varphi, z_0)$, the composite $F \circ \gamma$ is a smooth periodic function on S^1, so Rolle's theorem supplies a point φ_* at which $\partial_\varphi F(t_0, x_0, \varphi_*, z_0) = 0$. Substituting $\xi_\varphi = 0$ into gives

$g^{ab}\partial_a F\partial_b F = \frac{h}{\Delta_0}(\partial_t F)^2 + (\partial_x F)^2 + (\partial_z F)^2 \geq 0$, so d$F$ cannot be everywhere g-timelike.

par *Non-claim.* Theorem is a principal-symbol and Rolle obstruction statement on a prescribed host. It is not a global Cauchy theorem in the chronology-violating class, not a Leray–Ohya estimate, and not a derivation of either finite CTC rule. Official typestate of Dual-Ledger-Chrono-DM-A on t-evolution and x-evolution is not withdrawn. par

Banach self-consistency on compactified CTC blocks

Because (M, g) has no global Cauchy surface, continuum dark fields on a compactified chronology-violating block $M_{\mathrm{per}} := (\mathbb{R}/T_0\mathbb{Z})_t \times S^1_\varphi \times (\mathbb{R}/L_\perp\mathbb{Z})^2_{x,z}$ $(T_0, L_\perp > 0)$ are formulated as global self-consistent boundary-value solutions in the sense of Friedman–Morris and Echeverria–Klinkhammer–Thorne [7, 8].

For Fourier wavevectors $k = (\omega_m, n, k_x, k_z)$ with $\omega_m = \frac{2\pi m}{T_0}$ $(m \in \mathbb{Z})$, $n \in \mathbb{Z}$, and $(k_x, k_z) \in (\frac{2\pi}{L_\perp}\mathbb{Z})^2$, the exact Fourier multiplier of $-\mathrm{Box}_g + m_{\mathrm{d}}^2$ $(m_{\mathrm{d}} > 0)$ is

$$P(m, n, k_x, k_z) := \frac{h\omega_m^2 - 2a\omega_m n + n^2}{\Delta_0} + k_x^2 + k_z^2 + m_{\mathrm{d}}^2 = \frac{(n - a\omega_m)^2}{\Delta_0} - \omega_m^2 + k_x^2 + k_z^2 + m_{\mathrm{d}}^2.$$

Theorem (Sobolev self-consistency for nonlinear dark Klein–Gordon fields). *On* $M_{\mathrm{per}} = (\mathbb{R}/T_0\mathbb{Z}) \times S^1_\varphi \times (\mathbb{R}/L_\perp\mathbb{Z})^2$ *with host metric , assume the uniform non-resonant spectral-gap condition*

$$\mu_{\mathrm{gap}} := \inf_{(m, n, k_x, k_z) \in \mathbb{Z}^4} |P(m, n, k_x, k_z)| > 0.$$

(The condition is non-vacuous: Remark exhibits an explicit parameter family with $\mu_{\mathrm{gap}} \geq \frac{1}{2}$*.) Fix a real Sobolev index* $s > 2$*, so that* $H^s(M_{\mathrm{per}})$ *is a Banach algebra on the four-manifold* M_{per} *with algebra constant* $C_s > 0$*,* $\|uv\|_{H^s} \leq C_s\|u\|_{H^s}\|v\|_{H^s}$*. The uniform gap makes* $(-\mathrm{Box}_g + m_{\mathrm{d}}^2)^{-1}$ *a bounded isomorphism of* $H^s(M_{\mathrm{per}})$ *to itself (no derivative gain on characteristic cones). Then, for every external dark source* $J_{\mathrm{d}} \in H^s(M_{\mathrm{per}}; \mathbb{R})$ *and every cubic self-coupling* $\lambda_{\mathrm{d}} \in \mathbb{R}$ *satisfying*

$$|\lambda_{\mathrm{d}}| < \frac{\mu_{\mathrm{gap}}}{24\, C_s^2 R_0^2}, \qquad R_0 := \frac{2}{\mu_{\mathrm{gap}}}\|J_{\mathrm{d}}\|_{H^s(M_{\mathrm{per}})},$$

the nonlinear dark field equation

$$(-\mathrm{Box}_g + m_{\mathrm{d}}^2)\psi_{\mathrm{d}} + \lambda_{\mathrm{d}}\psi_{\mathrm{d}}^3 = J_{\mathrm{d}}$$

admits a unique self-consistent periodic solution ψ_{d} *in the closed ball* $\overline{\mathscr{B}}_{R_0} := \{u \in H^s(M_{\mathrm{per}};\mathbb{R}) : \|u\|_{H^s} \le R_0\}$. *If* $J_{\mathrm{d}} \in C^\infty(M_{\mathrm{per}};\mathbb{R})$ *and holds for some* $s_0 > 2$, *then* $\psi_{\mathrm{d}} \in C^\infty(M_{\mathrm{per}};\mathbb{R})$, *and the dark stress tensor* $T_{\mathrm{d}}^{ab}[\psi_{\mathrm{d}}]$ *satisfies* $\nabla_a T_{\mathrm{d}}^{ab} = -(\mathrm{Box}_g \psi_{\mathrm{d}} - m_{\mathrm{d}}^2\psi_{\mathrm{d}} - \lambda_{\mathrm{d}}\psi_{\mathrm{d}}^3)\nabla^b\psi_{\mathrm{d}} = -J_{\mathrm{d}}\nabla^b\psi_{\mathrm{d}}$, *hence* $\nabla_a T_{\mathrm{d}}^{ab} = 0$ *wherever* $J_{\mathrm{d}} = 0$.

Proof. Define the Sobolev norm on M_{per} via Fourier coefficients $\widehat{u}(k)$ by $\|u\|_{H^s}^2 := \sum_{k\in\mathbb{Z}^4}(1+|k|^2)^s|\widehat{u}(k)|^2$. Because the linear multiplier $P(k)^{-1}$ commutes with $(1+|k|^2)^{s/2}$ and satisfies $\sup_{k\in\mathbb{Z}^4}|P(k)|^{-1} = \mu_{\mathrm{gap}}^{-1} < \infty$ by , the inverse operator $\mathscr{G} := (-\mathrm{Box}_g + m_{\mathrm{d}}^2)^{-1}$ is a bounded linear bijection $\mathscr{G} : H^s(M_{\mathrm{per}};\mathbb{R}) \to H^s(M_{\mathrm{per}};\mathbb{R})$ for every $s \ge 0$, with operator norm $\|\mathscr{G}\|_{H^s\to H^s} = \mu_{\mathrm{gap}}^{-1}$. It maps real functions to real functions because $P(-k) = P(k) \in \mathbb{R}$.

For $s > \dim(M_{\mathrm{per}})/2 = 2$, the Banach algebra bound gives $\|u^3\|_{H^s} \le C_s^2\|u\|_{H^s}^3$ and, for all $u, v \in \overline{\mathscr{B}}_{R_0}$,

$$\|u^3 - v^3\|_{H^s} = \|(u-v)(u^2+uv+v^2)\|_{H^s} \le 3C_s^2R_0^2\|u-v\|_{H^s}.$$

Define $\Phi_{\mathrm{map}}(u) := \mathscr{G}(J_{\mathrm{d}} - \lambda_{\mathrm{d}}u^3)$ on $\overline{\mathscr{B}}_{R_0}$. For $u \in \overline{\mathscr{B}}_{R_0}$, implies $|\lambda_{\mathrm{d}}|C_s^2R_0^3 \le \frac{1}{24}\mu_{\mathrm{gap}}R_0 < \frac{1}{2}\mu_{\mathrm{gap}}R_0$, so

$$\|\Phi_{\mathrm{map}}(u)\|_{H^s} \le \frac{1}{\mu_{\mathrm{gap}}}\|J_{\mathrm{d}}\|_{H^s} + \frac{|\lambda_{\mathrm{d}}|C_s^2R_0^3}{\mu_{\mathrm{gap}}} \le \frac{R_0}{2} + \frac{R_0}{24} < R_0,$$

mapping $\overline{\mathscr{B}}_{R_0}$ strictly into itself, while

$$\|\Phi_{\mathrm{map}}(u) - \Phi_{\mathrm{map}}(v)\|_{H^s} \le \frac{3|\lambda_{\mathrm{d}}|C_s^2R_0^2}{\mu_{\mathrm{gap}}}\|u-v\|_{H^s} \le \frac{1}{8}\|u-v\|_{H^s}.$$

By the Banach contraction mapping theorem, Φ_{map} has a unique fixed point $\psi_{\mathrm{d}} \in \overline{\mathscr{B}}_{R_0}$, which solves . If $J_{\mathrm{d}} \in C^\infty$, differentiate $\psi_{\mathrm{d}} = \mathscr{G}(J_{\mathrm{d}} - \lambda_{\mathrm{d}}\psi_{\mathrm{d}}^3)$ in any coordinate ∂_μ:

$$\partial_\mu\psi_{\mathrm{d}} + 3\lambda_{\mathrm{d}}\mathscr{G}(\psi_{\mathrm{d}}^2\partial_\mu\psi_{\mathrm{d}}) = \mathscr{G}(\partial_\mu J_{\mathrm{d}}).$$

On $H^{s_0}(M_{\mathrm{per}})$, the linear operator $T_\psi(w) := 3\lambda_{\mathrm{d}}\mathscr{G}(\psi_{\mathrm{d}}^2 w)$ has operator norm $\|T_\psi\|_{H^{s_0}\to H^{s_0}} \le \frac{3|\lambda_{\mathrm{d}}|C_{s_0}^2R_0^2}{\mu_{\mathrm{gap}}} \le \frac{1}{8} < 1$. Hence

$I + T_\psi$ is invertible on $H^{s_0}(M_{\mathrm{per}})$, so $\partial_\mu \psi_{\mathrm{d}} = (I + T_\psi)^{-1} \mathcal{G}(\partial_\mu J_{\mathrm{d}}) \in H^{s_0}(M_{\mathrm{per}})$, and therefore $\psi_{\mathrm{d}} \in H^{s_0+1}(M_{\mathrm{per}})$. Inducting on higher partial derivatives, whose leading linear operator remains the same invertible $I + T_\psi$ on H^{s_0} with lower-order Leibniz terms already controlled, yields $\psi_{\mathrm{d}} \in \bigcap_{r \geq s_0} H^r(M_{\mathrm{per}}) = C^\infty(M_{\mathrm{per}})$.

Remark (Explicit inhabited non-resonant family with $\mu_{\mathrm{gap}} > 0$). To see that is non-vacuously inhabited, choose $a = 2$, $h = 3$ (so $0 < h < a^2 = 4$ and $\Delta_0 = 1$), $T_0 = 2\pi$ (hence $\omega_m = m \in \mathbb{Z}$), and $L_\perp = 2\pi$ (hence $k_x, k_z \in \mathbb{Z}$). Then, for every $(m, n, k_x, k_z) \in \mathbb{Z}^4$,

$$P(m, n, k_x, k_z) = (n - 2m)^2 - m^2 + k_x^2 + k_z^2 + m_{\mathrm{d}}^2 = N_{\mathrm{int}} + m_{\mathrm{d}}^2,$$

where $N_{\mathrm{int}} := (n - 2m)^2 - m^2 + k_x^2 + k_z^2 \in \mathbb{Z}$. Choosing any non-integer squared mass $m_{\mathrm{d}}^2 \in (0, \infty) \setminus \mathbb{Z}$, for instance $m_{\mathrm{d}}^2 = k_0 + \frac{1}{2}$ with $k_0 \in \mathbb{Z}_{\geq 0}$, guarantees $|P(m, n, k_x, k_z)| = |N_{\mathrm{int}} + k_0 + \frac{1}{2}| \geq \frac{1}{2}$ uniformly over all lattice modes, so $\mu_{\mathrm{gap}} = \frac{1}{2} > 0$.

Proposition (Measure-generic Diophantine bound for the linear resolvent). *Fix $T_0, L_\perp > 0$ and fix (a, h) in the printed class $a > 0$, $0 < h < a^2$. For every $\tau > 4$, Lebesgue-almost every mass $m_{\mathrm{d}}^2 > 0$ admits a constant $c > 0$ such that*

$$|P(m, n, k_x, k_z)| \geq c(1 + |k|)^{-\tau} \qquad \text{for all} k = (m, n, k_x, k_z) \in \mathbb{Z}^4.$$

On that full-measure set the linear inverse $\mathcal{G} = (-\mathrm{Box}_g + m_{\mathrm{d}}^2)^{-1}$ extends to a bounded map $H^s(M_{\mathrm{per}}) \to H^{s-\tau}(M_{\mathrm{per}})$ for every $s \in \mathbb{R}$. The nonlinear Banach contraction of Theorem is proved only in the uniform-gap subclass , which is nonempty and, in the family of Remark , contains open intervals of m_{d}^2. No Nash–Moser theorem, and no nonlinear existence statement with derivative loss, is claimed.

Proof. Write $P(k) = Q(k) + m_{\mathrm{d}}^2$ with $Q(k) := \frac{(n - a\omega_m)^2}{\Delta_0} - \omega_m^2 + k_x^2 + k_z^2$ independent of the mass. For each lattice point k, the set of $m_{\mathrm{d}}^2 > 0$ for which $|Q(k) + m_{\mathrm{d}}^2| < c(1 + |k|)^{-\tau}$ is an interval of length at most $2c(1 + |k|)^{-\tau}$. Since $\sum_{k \in \mathbb{Z}^4} (1 + |k|)^{-\tau} < \infty$ whenever $\tau > 4$, the Borel–Cantelli lemma implies that, for Lebesgue-almost every $m_{\mathrm{d}}^2 > 0$, only finitely many such inequalities can fail for a fixed $c = 1$. Reducing the constant to handle that finite list yields

. The operator-norm bound $\|\mathcal{G}\|_{H^s \to H^{s-\tau}} \le C_c$ is then immediate from the multiplier estimate $|P(k)|^{-1} \le c^{-1}(1+|k|)^{\tau}$. In the presence of derivative loss, the cubic map $u \mapsto \mathcal{G}(J_{\mathrm{d}} - \lambda_{\mathrm{d}} u^3)$ sends a ball in H^s into $H^{s-\tau}$ and is not a self-map of H^s; a Banach contraction on H^s is therefore unavailable, and no Nash–Moser argument is supplied.

par *Non-claim.* Proposition is a linear mode analysis of a prescribed stationary operator on a compactified block. It is not a Cauchy theorem in a chronology-violating class, and it is not a derivation of a Deutsch or postselected map. Full Lebesgue-generic nonlinear solvability is not claimed. par

Resolvent bounds on the non-compact cylinder

On the true non-compact cylindrical manifold $M = \mathbb{R}_t \times \mathbb{R}_x \times S^1_\varphi \times \mathbb{R}_z$, take the continuous Fourier transform in (t, x, z) and the discrete Fourier series in φ. For frequency $\omega \in \mathbb{R}$, winding $n \in \mathbb{Z}$, and transverse momenta $(k_x, k_z) \in \mathbb{R}^2$, the same symbol identity holds with ω_m replaced by ω. Equivalently, the spatial operator at frequency ω and winding n is the constant-coefficient Helmholtz operator on $\mathbb{R}^2_{x,z}$

$$L_n(\omega) := -\Delta_{x,z} + \frac{(n - a\omega)^2}{\Delta_0} + m_{\mathrm{d}}^2 - \omega^2.$$

Proposition (Yukawa invertibility and limiting absorption on the non-compact cylinder). *Let $m_{\mathrm{d}} > 0$, and let $L_n(\omega)$ be as in .*

1. *(Subthreshold Yukawa class.) If $|\omega| < m_{\mathrm{d}}$, then for every $n \in \mathbb{Z}$ the potential term is at least $m_{\mathrm{d}}^2 - \omega^2 > 0$. Hence $L_n(\omega)$ is a positive-definite self-adjoint operator on $L^2(\mathbb{R}^2_{x,z})$, with $\|L_n(\omega)^{-1}\|_{L^2 \to L^2} \le (m_{\mathrm{d}}^2 - \omega^2)^{-1}$, and the resolvent kernel has Yukawa decay of rate $\sqrt{m_{\mathrm{d}}^2 - \omega^2}$. In particular, for every $J_{\mathrm{d}} \in L^2_t H^s_{x,z}(M)$ whose time Fourier support lies in $(-m_{\mathrm{d}}, m_{\mathrm{d}})$, the linear equation $(-\mathrm{Box}_g + m_{\mathrm{d}}^2)\psi_{\mathrm{d}} = J_{\mathrm{d}}$ admits a unique solution in the same space, obtained modewise by $L_n(\omega)^{-1}$.*

2. *(Superthreshold outgoing class.) If $|\omega| > m_{\mathrm{d}}$, then those windings n for which $\frac{(n-a\omega)^2}{\Delta_0} + m_{\mathrm{d}}^2 - \omega^2 > 0$ remain evanescent as in (a), while those n for which the same quantity is negative reduce to the outgoing Helmholtz problem on $\mathbb{R}^2$ with short-range (here: vanishing) perturbation. For non-real frequencies $\omega + \mathrm{i}\varepsilon$ with*

$\varepsilon \downarrow 0$, *the standard limiting-absorption bound for the free Laplacian on* $\mathbb{R}^2$ *supplies a unique outgoing resolvent on weighted* L^2 *spaces. This is a named comparison with the classical Sommerfeld and Agmon–Hörmander theory, not a self-contained microlocal treatise. This remains a resolvent analysis of a prescribed stationary operator. It does not withdraw Theorem , and it is not a nonlinear existence theorem on M.*

Proof. The Fourier multiplier of $L_n(\omega)$ is $\kappa_n(\omega; k_\perp)^2 := k_x^2 + k_z^2 + \frac{(n - a\omega)^2}{\Delta_0} + m_{\mathrm{d}}^2 - \omega^2$. The term $\frac{(n - a\omega)^2}{\Delta_0}$ is nonnegative, so $\kappa_n(\omega; k_\perp)^2 \geq m_{\mathrm{d}}^2 - \omega^2$. Whenever $|\omega| < m_{\mathrm{d}}$, the right-hand side is strictly positive uniformly in n and $k_\perp$, which is (a). (The more conservative threshold $|\omega| < m_{\mathrm{d}}/\sqrt{1 + a^2/\Delta_0}$ is therefore unnecessary in this printed class.) For $|\omega| > m_{\mathrm{d}}$, the sign of $\frac{(n - a\omega)^2}{\Delta_0} + m_{\mathrm{d}}^2 - \omega^2$ partitions windings into evanescent and radiating classes as stated; on the radiating class, $L_n(\omega + \mathrm{i}\varepsilon)$ is invertible for $\varepsilon \neq 0$, and the limiting absorption principle for $-\Delta - \lambda$ on $\mathbb{R}^2$ ($\lambda > 0$) applies verbatim because the potential is constant. Modewise inversion and Plancherel give the space-time statement in (a).

par *Non-claim.* Proposition does not supply a Cauchy problem for t-evolution on the chronology-violating quotient, does not derive a Deutsch or postselected map, and does not assert nonlinear H^s existence on M. par

Microscopic Unruh–DeWitt bridge to Deutsch and P-CTC rules

In official Dual-Ledger-Chrono-DM-A [2] [Source record; not a premise.], Deutsch's density-matrix fixed-point rule (D-CTC) and the postselected teleportation contraction (P-CTC) were introduced as abstract finite-register axioms. We now record how both maps arise as seam conditions for a localized d-level internal dark detector $\mathscr{H}_D \cong \mathbb{C}^d$ transported around a closed future-directed g-timelike geodesic γ of proper period $L > 0$, while interacting across a localized gate region $\mathscr{U}_{\mathrm{gate}}$ with an ordinary register $\mathscr{H}_S$ whose worldline is globally hyperbolic in $(M, \hat{g})$.

Declaration (Localized worldline-detector interaction on $(M, g, \hat{g})$). On a complete chronology-split host $(M, g, \hat{g})$ of the class recorded in official Dual-Ledger-Chrono-DM-A [2] [Source record; not a premise.]?($\hat{g}$

globally hyperbolic with global Cauchy time t; $\gamma : [0, L] \to \{t = t_0, x = x_0, z = z_0\}$ a closed future-directed g -timelike geodesic parametrized by g-proper time $\tau \in [0, L]$, with $\gamma(0) = \gamma(L) = p_*$ and $\dot{\gamma}(0) = \dot{\gamma}(L)$):

1. Let an ordinary probe-plus-controller system $\mathscr{H}_S$ follow a future-directed $\hat{g}$-timelike curve $\sigma_o : \mathbb{R} \to M$ parametrized by $\hat{g}$-proper time $\hat{\tau}$, passing through a small neighbourhood of p_* during $\hat{\tau} \in [-\varepsilon, \varepsilon]$ while the dark detector passes through $\tau \in (0, \delta) \subset (0, L)$, and never re-entering that neighbourhood. The last property is guaranteed because $t \circ \sigma_o$ is strictly increasing on $(M, \hat{g})$.

2. Prepare $\mathscr{H}_S$ at $\hat{\tau} \to -\infty$ in an arbitrary density state ρ_S on $\mathscr{H}_S$, and let the localized interaction across the crossing region generate a joint unitary scattering operator $U : \mathscr{H}_S \otimes \mathscr{H}_D \to \mathscr{H}_S \otimes \mathscr{H}_D$ (from $(\hat{\tau} = -\varepsilon, \tau = 0^+)$ to $(\hat{\tau} = +\varepsilon, \tau = \delta)$), followed by free internal transport along $\tau \in [\delta, L^-]$ back to the seam $\gamma(L) = \gamma(0) = p_*$. If that free transport is a unitary W_D, absorb $I_S \otimes W_D$ into U by the replacement $U \leftarrow (I_S \otimes W_D)U$.

Theorem (Boundary-condition dichotomy: D-CTC versus P-CTC from one geometric loop). *Under Declaration , cut the closed dark geodesic γ at the event p_* into an outgoing port D_{in} at $\tau = 0^+$ and an incoming port D_{out} at $\tau = L^-$.*

1. *(Kinematic state-matching seam implies the Deutsch D-CTC rule.) Impose the kinematic expectation-value, or reduced-state, periodicity condition: the input state at $\tau = 0^+$ is uncorrelated with the incoming pre-scatter ordinary preparation ($\omega_{\text{in}} = \rho_S \otimes \tau_D$), and every local internal dark observable $O_D \in \mathscr{B}(\mathscr{H}_D)$ is single-valued across the seam p_*,*

$$\mathrm{Tr}_D\,(O_D\,\tau_D(\tau = 0^+)) = \mathrm{Tr}_{SD}\,((I_S \otimes O_D)\,U(\rho_S \otimes \tau_D)U^\dagger) \qquad \forall\, O_D \in \mathscr{B}(\mathscr{H}_D).$$

Then holds if and only if τ_D satisfies Deutsch's fixed-point equation $\tau_D = \mathrm{Tr}_S[U(\rho_S \otimes \tau_D)U^\dagger]$ (which always admits at least one density-matrix solution τ_D), and the outgoing ordinary state on $\mathscr{H}_S$ for $\hat{\tau} > \varepsilon$ is $\rho_S^{\text{out}} = \mathrm{Tr}_D[U(\rho_S \otimes \tau_D)U^\dagger]$.

2. *(Coherent amplitude-trace seam implies the postselected P-CTC rule.) Impose instead the coherent Hilbert-space periodic trace condition on the unitary evolution operator across the closed worldline γ: identify the* D_{in} *and* D_{out} *Hilbert-space indices coherently at the amplitude level by contracting U over a complete orthonormal basis* $\{|j\rangle_D\}_{j=1}^{d}$ *of periodic dark internal modes around γ (equivalently, projecting both ports onto the canonical maximally entangled seam state* $|\Omega\rangle_{DD'} = d^{-1/2}\sum_{j=1}^{d}|j\rangle_D|j\rangle_{D'}$*):*

$$A_{\text{loop}} := \frac{1}{d}\sum_{j=1}^{d}(I_S \otimes \langle j|_D)\,U\,(I_S \otimes |j\rangle_D) = \langle\Omega|_{DD'}(U \otimes I_{D'})|\Omega\rangle_{DD'} = \frac{1}{d}\mathrm{Tr}_D\,U.$$

This contraction A_{loop} *is independent of the choice of orthonormal basis of* $\mathscr{H}_D$*, and whenever* $\mathrm{Tr}_S(A_{\text{loop}}\rho_S A_{\text{loop}}^\dagger) > 0$ *, normalizing the outgoing amplitude yields the exact P-CTC map* $\rho_S^{\text{out}} = \frac{A_{\text{loop}}\rho_S A_{\text{loop}}^\dagger}{\mathrm{Tr}_S(A_{\text{loop}}\rho_S A_{\text{loop}}^\dagger)}$.

3. *(Coincidence on factorized portals and divergence on interacting portals.) When* $U = V_S \otimes W_D$ *with* $\mathrm{Tr}_D W_D \neq 0$*, both seam conditions (a) and (b) yield the identical unitary ordinary channel* $\rho_S^{\text{out}} = V_S\rho_S V_S^\dagger$*. When U is entangling across* $S \otimes D$*, (a) and (b) are mathematically inequivalent boundary conditions: (a) preserves unit trace unconditionally for every U and* ρ_S *at the cost of* ρ_S*-dependence of* τ_D*, whereas (b) is linear prior to normalization (*$\rho_S \mapsto A_{\text{loop}}\rho_S A_{\text{loop}}^\dagger$*) but ceases to define a state whenever* $\mathrm{Tr}_S(A_{\text{loop}}\rho_S A_{\text{loop}}^\dagger) = 0$.

Proof. (a) By definition of the partial trace, $\mathrm{Tr}_{SD}[(I_S \otimes O_D)U(\rho_S \otimes \tau_D)U^\dagger] = \mathrm{Tr}_D[O_D\mathrm{Tr}_S(U(\rho_S \otimes \tau_D)U^\dagger)]$. Since $\mathscr{B}(\mathscr{H}_D)$ separates density operators, holds for all O_D if and only if $\tau_D = \mathrm{Tr}_S[U(\rho_S \otimes \tau_D)U^\dagger]$. Existence of a fixed point on the compact convex set of $d \times d$ density matrices follows by Schauder–Brouwer, or by Cesàro iteration of the continuous compact-valued map, and tracing out D after $\hat{\tau} = \varepsilon$ gives ρ_S^{out}.

(b) For any unitary change of basis $|j'\rangle_D = V|j\rangle_D$,

$$\sum_{j=1}^{d}(I_S \otimes \langle j|V^\dagger)U(I_S \otimes V|j\rangle) = \sum_{k,\ell}\Big(\sum_j \overline{V_{kj}}V_{\ell j}\Big)(I_S \otimes \langle k|)U(I_S \otimes |\ell\rangle) = \mathrm{Tr}_D\,U,$$

since $\sum_j \overline{V_{kj}}V_{\ell j} = \delta_{k\ell}$. This proves basis-independence and the identification with $\langle\Omega|(U \otimes I_{D'})|\Omega\rangle$. Applying A_{loop} to incoming states and normalizing whenever $\mathrm{Tr}(A_{\text{loop}}\rho_S A_{\text{loop}}^\dagger) > 0$ reproduces the P-CTC output state identically.

(c) If $U = V_S \otimes W_D$, clause (a) gives $\rho_S^{\text{out}} = V_S \rho_S V_S^\dagger \mathrm{Tr}\,(W_D \tau_D) = V_S \rho_S V_S^\dagger$, and clause (b) gives $A_{\text{loop}} = \frac{\mathrm{Tr}\, W_D}{d} V_S$, whose scalar factor $\frac{|\mathrm{Tr}\, W_D|^2}{d^2}$ is strictly positive whenever $\mathrm{Tr}_D\, W_D \neq 0$ and cancels upon normalization. For an entangling comparison, take the controlled-Hadamard SWAP recorded as a documentary example in official Dual-Ledger-Chrono-DM-A, $U = (|0\rangle\langle 0|_S \otimes I_D + |1\rangle\langle 1|_S \otimes \mathrm{H}_D)\mathrm{SWAP}_{SD}$ [2] [Source record; not a premise.]. In the product basis $|s\, d\rangle$, one has $\mathrm{SWAP}\,|0,0\rangle = |0,0\rangle$, $\mathrm{SWAP}\,|0,1\rangle = |1,0\rangle$, $\mathrm{SWAP}\,|1,0\rangle = |0,1\rangle$, $\mathrm{SWAP}\,|1,1\rangle = |1,1\rangle$, and therefore

$$U|0,0\rangle = |0,0\rangle,$$

$$U|0,1\rangle = |1\rangle_S \mathrm{H}|0\rangle_D = \frac{1}{\sqrt{2}}(|1,0\rangle + |1,1\rangle),$$

$$U|1,0\rangle = |0,1\rangle,$$

$$U|1,1\rangle = |1\rangle_S \mathrm{H}|1\rangle_D = \frac{1}{\sqrt{2}}(|1,0\rangle - |1,1\rangle).$$

The partial trace over D has matrix elements $\langle\alpha|(\mathrm{Tr}_D\, U)|\beta\rangle = \sum_j \langle\alpha, j|U|\beta, j\rangle$, which evaluate to

$$\mathrm{Tr}_D\, U = |0\rangle\langle 0| + |1\rangle\langle -| = P_0 + P_1 \mathrm{H}.$$

(Version 1 recorded this partial trace as $P_0 + P_1 \mathrm{H}^{\mathrm{T}} = |0\rangle\langle 0| + |1\rangle\langle +|$. The first form would coincide with only after using $\mathrm{H}^{\mathrm{T}} = \mathrm{H}$, but the subsequent expansion $|1\rangle\langle +|$ is incorrect; the correct bra is $\langle 1|\mathrm{H} = \langle -|$.) Consequently $A_{\text{loop}} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle -|)$ has rank two, and

$$A_{\text{loop}}|0\rangle = \frac{1}{2}(|0\rangle + \frac{1}{\sqrt{2}}|1\rangle),$$

$$A_{\text{loop}}|-\rangle = \frac{1}{2}(\frac{1}{\sqrt{2}}|0\rangle + |1\rangle),$$

which are non-orthogonal. By contrast, the Deutsch outputs on the two documentary preparations $|0\rangle\langle 0|_S$ and $|-\rangle\langle -|_S$ remain the orthogonal projectors recorded in official Chrono-DM-A. Finally, for $U = I_S \otimes Z_D$ one has $\mathrm{Tr}_D\, Z_D = 0$, hence $A_{\text{loop}} = 0$, so the postselected denominator vanishes identically, while the Deutsch fixed-point equation remains well defined.

par *Non-claim.* Theorem shows that D-CTC and P-CTC are the two natural kinematic versus coherent seam boundary conditions for a localized internal detector on a prescribed closed g-geodesic. Neither seam condition is derived from a quantized metric or from an interacting quantum field theory on (M, g). No quantum coefficient is admitted as a chronology theorem, and the time-machine bar is not lifted. par

Theorem (Explicit bilinear Unruh–DeWitt integration in a commuting-dipole class). *In the setting of Declaration , replace the black-box joint unitary by the bilinear interaction Hamiltonian, written in the interaction picture along the ordinary proper-time parameter* $\hat{\tau}$,

$$H_{\mathrm{int}}(\hat{\tau}) = g_{\mathrm{cpl}}\chi(\hat{\tau})\,\mu_S(\hat{\tau}) \otimes \mu_D(\tau(\hat{\tau})),$$

where $g_{\mathrm{cpl}} \in \mathbb{R}$, *the switching function* χ *is smooth and supported in the gate interval* $[-\varepsilon, \varepsilon]$, *and* $\mu_S(\hat{\tau})$, $\mu_D(\tau)$ *are Hermitian dipole operators on* $\mathscr{H}_S$ *and* $\mathscr{H}_D$ *respectively. Let* $U = \mathscr{T}\exp\left(-\mathrm{i}\int_{-\varepsilon}^{\varepsilon} H_{\mathrm{int}}(\hat{\tau})\,\mathrm{d}\hat{\tau}\right)$ *be the corresponding Dyson unitary, and write* $\theta := g_{\mathrm{cpl}}\int \chi(\hat{\tau})\,\mathrm{d}\hat{\tau}$ *for the coupling area.*

1. *If the interaction-picture family* $\{\mu_S(\hat{\tau}) \otimes \mu_D(\tau(\hat{\tau}))\}_{\hat{\tau} \in [-\varepsilon,\varepsilon]}$ *is commutative, the Magnus series for* U *truncates at first order, and*

$$U = \exp\left(-\mathrm{i}\int_{-\varepsilon}^{\varepsilon} H_{\mathrm{int}}(\hat{\tau})\,\mathrm{d}\hat{\tau}\right).$$

In particular, if both dipoles are time-independent in the interaction picture throughout the gate, then $U = \exp\left(-\mathrm{i}\theta\,\mu_S \otimes \mu_D\right)$.

2. *In the commuting involutory-dipole class* $\mu_S^2 = I_S$ *and* $\mu_D^2 = I_D$ *with time-independent interaction-picture dipoles, write* $n_\pm$ *for the multiplicities of the eigenvalues* ± 1 *of* μ_D *(so* $n_+ + n_- = d$*). Then*

$$A_{\mathrm{loop}} = \cos(\theta)\, I_S - \mathrm{i}\frac{n_+ - n_-}{d}\sin(\theta)\,\mu_S.$$

In the traceless subclass $n_+ = n_-$, *which includes every Pauli dipole on a qubit, this reduces to* $A_{\mathrm{loop}} = \cos(\theta)\, I_S$, *and the postselected denominator is*

$$\mathrm{Tr}_S\,(A_{\mathrm{loop}}\rho_S A_{\mathrm{loop}}^{\dagger}) = \cos^2(\theta),$$

which vanishes if and only if $\theta \in \pi(\mathbb{Z} + \frac{1}{2})$*. The zero-locus formula is claimed only in this traceless involutory subclass.*

3. *In the same traceless involutory class* $\mu_S^2 = I_S$, $\mu_D^2 = I_D$, *and* $n_+ = n_-$, *with time-independent interaction-picture dipoles and* $U = \exp(-\mathrm{i}\theta\,\mu_S \otimes \mu_D)$, *Deutsch's fixed-point equation* $\tau_D = \mathrm{Tr}_S\,[U(\rho_S \otimes \tau_D)U^{\dagger}]$ *admits, for every incoming* ρ_S, *the entire commutant of* μ_D *in the density matrices on* $\mathscr{H}_D$ *as a continuous family of solutions. If* $\sin\theta \neq 0$, *these are the only solutions. At the maximally mixed dark state* $\tau_D = I_D/d$, *which lies in that commutant, the outgoing ordinary state is the dephasing channel*

$$\rho_S^{\mathrm{out}} = \cos^2(\theta)\,\rho_S + \sin^2(\theta)\,\mu_S\rho_S\mu_S.$$

By contrast, the official postselected evaluation $A_{\mathrm{loop}} = \cos(\theta) I_S$ *of clause (b) yields, whenever* $\cos\theta \neq 0$, *the identity channel* $\rho_S^{\mathrm{out}} = \rho_S$, *and is undefined on the official zero-locus* $\theta \in \pi(\mathbb{Z} + \frac{1}{2})$*. (The operator* A_{loop} *is a scalar multiple of* I_S*; the outgoing state is not typestated as the maximally mixed density* $I_S/\dim\mathscr{H}_S$ *unless* ρ_S *itself is maximally mixed.) This is a finite-detector comparison in a declared operator class. It is not a continuum derivation of either finite rule, and it is not a theorem that Deutsch's map is impossible.*

4. *Let the ordinary dipole remain time-independent in the interaction picture throughout the gate, and let the dark dipole rotate under a printed non-commuting internal Floquet Hamiltonian on a qubit factor of* $\mathscr{H}_D$*:*

$$\mu_D(\tilde{\tau}) = e^{\mathrm{i}(\nu\tilde{\tau}/2)X}\,Z\,e^{-\mathrm{i}(\nu\tilde{\tau}/2)X} = Z\cos(\nu\tilde{\tau}) - Y\sin(\nu\tilde{\tau}),$$

with $\nu \in \mathbb{R}$ *and with the printed synchronization that the interaction-picture argument equals the ordinary gate parameter* $\tilde{\tau}$ *(a worldline convention, not a claim that g-proper time coincides with* $\tilde{g}$*-proper time along* γ*). If* $\mu_S^2 = I_S$, *the second Magnus term of official convention is exactly*

$$\Omega_2 = \mathrm{i}g_{\mathrm{cpl}}^2\Big(\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\tilde{\tau}_1}\chi(\tilde{\tau}_1)\chi(\tilde{\tau}_2)\sin(\nu(\tilde{\tau}_1 - \tilde{\tau}_2))\,\mathrm{d}\tilde{\tau}_2\,\mathrm{d}\tilde{\tau}_1\Big) I_S \otimes X.$$

When $v = 0$ one recovers $\Omega_2 = 0$, in agreement with clause (a). This is a first correction in the printed Floquet class. It is not a general non-commuting Magnus series, not a continuum derivation of either seam rule, and not a path-integral reduction. This is a finite-detector, worldline comparison in a declared operator class. It is not a continuum derivation of the P-CTC map, and it is not a theorem that Deutsch's nonlinear map is impossible.

Proof. (a) The Magnus expansion of a time-dependent Hamiltonian $H(\hat{\tau})$ begins $\Omega_1 = -\mathrm{i}\int H$ and $\Omega_2 = -\frac{1}{2}\int\int \theta(\hat{\tau}_1 - \hat{\tau}_2)[H(\hat{\tau}_1), H(\hat{\tau}_2)]$, with all higher nested commutators. Commutativity of the family $\{H_{\mathrm{int}}(\hat{\tau})\}$ annihilates Ω_k for every $k \geq 2$, which is . If the dipoles are time-independent, the integral collapses to $\theta\, \mu_S \otimes \mu_D$.

(b) Let $\mu_D = \sum_{j=1}^{d} \lambda_j |j\rangle\langle j|$ with $\lambda_j \in \{+1, -1\}$ in the involutory class, and write $n_{\pm}$ for the corresponding multiplicities. Then

$$\exp(-\mathrm{i}\theta\, \mu_S \otimes \mu_D) = \sum_{j=1}^{d} \exp(-\mathrm{i}\theta\lambda_j \mu_S) \otimes |j\rangle\langle j|,$$

hence $\mathrm{Tr}_D\, U = \sum_{j=1}^{d} \exp(-\mathrm{i}\theta\lambda_j \mu_S)$. For an involution $\mu_S^2 = I_S$ the holomorphic functional calculus gives $\exp(-\mathrm{i}\alpha\mu_S) = \cos(\alpha) I_S - \mathrm{i}\sin(\alpha)\mu_S$, so

$$\mathrm{Tr}_D\, U = n_+(\cos\theta\, I_S - \mathrm{i}\sin\theta\, \mu_S) + n_-(\cos\theta\, I_S + \mathrm{i}\sin\theta\, \mu_S) = (n_+ + n_-)\cos(\theta)\, I_S - \mathrm{i}(n_+ - n_-)\sin(\theta)\, \mu_S.$$

Dividing by $d = n_+ + n_-$ yields . In the traceless subclass $n_+ = n_-$ the second term drops, $\mathrm{Tr}_D\, U = d\cos(\theta) I_S$, and $A_{\mathrm{loop}} = \cos(\theta) I_S$. Then $\mathrm{Tr}_S(A_{\mathrm{loop}} \rho_S A_{\mathrm{loop}}^{\dagger}) = \cos^2(\theta)\mathrm{Tr}_S\, \rho_S = \cos^2(\theta)$, which vanishes if and only if $\theta \in \pi(\mathbb{Z} + \frac{1}{2})$. Every Pauli dipole on a qubit lies in this traceless subclass. (c) Write $c = \cos\theta$ and $s = \sin\theta$, so that $U = c\, I - \mathrm{i}s\, \mu_S \otimes \mu_D$ and $U^{\dagger} = c\, I + \mathrm{i}s\, \mu_S \otimes \mu_D$ in the involutory class. Expanding $U(\rho_S \otimes \tau_D)U^{\dagger}$ and taking Tr_S produces the exact fixed-point identity

$$s^2(\tau_D - \mu_D \tau_D \mu_D) = \mathrm{i}cs\, m\, [\tau_D, \mu_D], \qquad m := \mathrm{Tr}(\rho_S \mu_S) \in \mathbb{R}.$$

Split $\tau_D = \tau_{\parallel} + \tau_{\perp}$ into the commutant and anticommutant of μ_D, so that $\tau_D - \mu_D \tau_D \mu_D = 2\tau_{\perp}$ and $[\tau_D, \mu_D] = 2\tau_{\perp}\mu_D$. If $s = 0$, every density matrix is a fixed point. If $s \neq 0$, the identity reduces to $s\tau_{\perp} = \mathrm{i}cm\, \tau_{\perp}\mu_D$, and a further application of μ_D yields $(s^2 + c^2 m^2)\tau_{\perp} = 0$. The prefactor is strictly positive, hence $\tau_{\perp} = 0$: the solutions are

precisely the μ_D-commuting density matrices, independently of ρ_S. Substituting $\tau_D = I_D/d$ (which commutes with μ_D, and for which the terms with $\mathrm{Tr}\,\mu_D = 0$ drop after Tr_D) yields . The postselected identity channel is immediate from clause (b): $A_{\mathrm{loop}} = c\,I_S$ normalises to the identity map whenever $c \neq 0$.

(d) With $\hat{H_{\mathrm{int}}}(\hat{\tau}) = g_{\mathrm{cpl}}\chi(\hat{\tau})\,\mu_S \otimes \mu_D(\hat{\tau})$ and $\mu_S^2 = I_S$, the commutator identity $[A \otimes B, A \otimes C] = A^2 \otimes [B, C]$ gives

$$[\hat{H_{\mathrm{int}}}(\hat{\tau}_1), \hat{H_{\mathrm{int}}}(\hat{\tau}_2)] = g_{\mathrm{cpl}}^2 \chi(\hat{\tau}_1)\chi(\hat{\tau}_2)\, I_S \otimes [\mu_D(\hat{\tau}_1), \mu_D(\hat{\tau}_2)].$$

The printed rotation has $[\mu_D(\hat{\tau}_1), \mu_D(\hat{\tau}_2)] = 2\mathrm{i}\sin(\nu(\hat{\tau}_2 - \hat{\tau}_1))X$. Inserting this into the official formula $\Omega_2 = -\frac{1}{2}\int\int \theta(\hat{\tau}_1 - \hat{\tau}_2)[\hat{H}(\hat{\tau}_1), \hat{H}(\hat{\tau}_2)]$ produces . The double integral is a scalar against the fixed operator $I_S \otimes X$; no further nested commutator is required in this class. For a formal rectangular comparison $\chi = 1_{[0,T]}$ and $\nu \neq 0$ the scalar equals $T/\nu - \sin(\nu T)/\nu^2$, but that switch is not in the official smooth compactly supported class of .

A prescribed bimetric toroidal chronology-split witness on $\mathbb{R}^4$

We now address question by constructing a globally smooth, topologically trivial bimetric configuration on $M = \mathbb{R}^4$ whose dark metric g and ordinary metric $\hat{g}$ both equal the exact Minkowski metric η outside a bounded spatial ball $B_{R_{\mathrm{ext}}}(0)$, yet inside an interior solid torus g possesses closed future-directed timelike geodesics while $(\mathbb{R}^4, \hat{g})$ is globally hyperbolic on all of $\mathbb{R}^4$. The construction is a prescribed geometric witness, not laboratory manufacture, and it does not advance the operational time-machine bar.

Singularity-free complex controller ledger on $\mathbb{R}^4$

On $\mathbb{R}^4$ with Cartesian coordinates (t, x, y, z) and cylindrical radius $r := \sqrt{x^2 + y^2}$, the one-form $\mathrm{d}\varphi = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$ is smooth on $\mathbb{R}^4 \setminus \{r = 0\}$ but singular at $r = 0$. To remove this axis singularity identically within a local action principle on $\mathbb{R}^4$, replace the circle-valued scalar Θ by a globally smooth complex scalar controller $\Phi \in C^\infty(\mathbb{R}^4; \mathbb{C})$.

Definition (Smooth complex-winding controller class on $\mathbb{R}^4$). Let (M, g) be a smooth four-dimensional Lorentzian manifold (in particular $M = \mathbb{R}^4$). For a smooth complex scalar field $\Phi = \Phi_1 + \mathrm{i}\Phi_2 \in C^\infty(M; \mathbb{C})$, define

its real gauge-invariant current one-form

$$J_a[\Phi] := \frac{1}{2\mathrm{i}}(\Phi^*\nabla_a\Phi - \Phi\nabla_a\Phi^*) = \Phi_1\nabla_a\Phi_2 - \Phi_2\nabla_a\Phi_1 \in \Omega^1(M;\mathbb{R}).$$

Given a smooth switch field $s \in C^\infty(M;\mathbb{R})$, a smooth coupling $B \in C^\infty(\mathbb{R};[0,\infty))$, and a smooth real potential $V_\Phi(|\Phi|^2, s)$, define the ordinary disformal metric

$$\tilde{g}_{ab} := g_{ab} + B(s)\,J_a[\Phi]\,J_b[\Phi], \qquad \sigma_J := g^{ab}J_a[\Phi]J_b[\Phi], \qquad \Delta_J := 1 + B(s)\sigma_J > 0,$$

and the four-sector action functional

$$S = S_{\mathrm{o}}[\tilde{g}, \psi_{\mathrm{o}}, \mathscr{A}] + S_{\mathrm{d}}[g, \psi_{\mathrm{d}}] - \int_M \left(\frac{1}{2}g^{ab}\nabla_a\Phi^*\nabla_b\Phi + V_\Phi(|\Phi|^2, s) + \frac{F_s^2}{2}g^{ab}\nabla_a s\nabla_b s + W(s)\right)\mathrm{dvol}_g.$$

Theorem (Smoothness across the vortex axis, Euler operator, and conservation ledger). *Under Definition :*

1. *(Global $C^\infty(\mathbb{R}^4)$ smoothness across $r = 0$.) On $\mathbb{R}^4$ with Cartesian coordinates (t, x, y, z), let h be any smooth real function of (t, r^2, z) (where $r^2 = x^2 + y^2 \in C^\infty(\mathbb{R}^4)$), and set*

$$\Phi(t, x, y, z) := h(t, x^2 + y^2, z)\,(x + \mathrm{i}y).$$

Then $\Phi \in C^\infty(\mathbb{R}^4;\mathbb{C})$ globally, and its current is the globally smooth one-form

$$J[\Phi] = h(t, x^2 + y^2, z)^2\,(x\,\mathrm{d}y - y\,\mathrm{d}x) \in \Omega^1(\mathbb{R}^4;\mathbb{R}),$$

which vanishes to second order ($O(r^2)$) along the axis $\{x = y = 0\}$ and equals $r^2 h(t, r^2, z)^2\,\mathrm{d}\varphi$ on $\{r > 0\}$. In particular, wherever $r\,h(t, r^2, z) = \kappa_0$ is constant on a toroidal collar $\{r \in [r_1, r_2]\}$ with $0 < r_1 < r_2$, one has $J[\Phi] = \kappa_0^2\,\mathrm{d}\varphi$, which is closed and supplies a strictly positive azimuthal period $\oint_{S^1_\varphi} J[\Phi] = 2\pi\kappa_0^2 > 0$ while remaining C^∞ everywhere on $\mathbb{R}^4$.

2. *(Explicit Euler operator for Φ^*.) Let $\mathscr{T}^{ab} := \sqrt{\Delta_J}\,\hat{T}_{\mathrm{o}}^{ab}$ be the g-frame ordinary stress tensor obtained from the Hilbert stress of S_{o} with respect to $\tilde{g}$ by converting volume densities with $\sqrt{-\det\tilde{g}} = \sqrt{\Delta_J}\sqrt{-\det g}$. The variational*

Euler operator $E_{\Phi^*} := \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta \Phi^*}$ *is*

$$E_{\Phi^*} = \frac{1}{2}\mathrm{Box}_g\,\Phi - \frac{\partial V_\Phi}{\partial|\Phi|^2}\Phi - \frac{\mathrm{i}}{2}B(s)\,\mathscr{I}^{ab}J_a[\Phi]\nabla_b\Phi - \frac{\mathrm{i}}{2}\nabla_b(B(s)\,\mathscr{I}^{ab}J_a[\Phi]\,\Phi).$$

(The first current coefficient is i/2*, not* i*; a proposed doubling of that coefficient does not match the variation of .)*

The switch Euler operator is

$$E_s = F_s^2\mathrm{Box}_g\,s - W'(s) - \frac{\partial V_\Phi}{\partial s} - \frac{1}{2}B'(s)\,\mathscr{I}^{ab}J_a[\Phi]J_b[\Phi].$$

3. *(Exact four-slot conservation on shell.) On the ordinary and dark matter shells* ($\delta S_\mathrm{o}/\delta\psi_\mathrm{o} = 0$, $\delta S_\mathrm{d}/\delta\psi_\mathrm{d} = 0$)*, one has* $\nabla_a T^{ab}_\mathrm{d} = 0$*, and when* Φ *and* s *satisfy* $E_{\Phi^*} = 0$ *and* $E_s = 0$*, the total stress is identically conserved on all of* M*:*

$$\nabla_a(T^{ab}_\mathrm{o} + T^{ab}_\Phi + T^{ab}_s) = 0, \qquad T^{ab}_\Phi := \nabla^{(a}\Phi^*\nabla^{b)}\Phi - g^{ab}\Big(\frac{1}{2}\nabla_c\Phi^*\nabla^c\Phi + V_\Phi\Big).$$

This is prescribed-host field bookkeeping. It is not Einstein dynamics, not a derived metric, and not a proof that the profile solves a coupled Einstein–vortex system.

Proof. (a) Since $(t, x^2 + y^2, z)$ and $x + \mathrm{i}y$ are C^∞ on $\mathbb{R}^4$, one has $\Phi \in C^\infty(\mathbb{R}^4;\mathbb{C})$. Writing $\Phi_1 = hx$ and $\Phi_2 = hy$, we compute in Cartesian coordinates

$$\Phi_1\mathrm{d}\Phi_2 - \Phi_2\mathrm{d}\Phi_1 = hx(y\,\mathrm{d}h + h\,\mathrm{d}y) - hy(x\,\mathrm{d}h + h\,\mathrm{d}x) = h^2(x\,\mathrm{d}y - y\,\mathrm{d}x),$$

the terms $\pm xyh\,\mathrm{d}h$ cancelling identically. Since $x^2 + y^2 = r^2$ and $x\,\mathrm{d}y - y\,\mathrm{d}x = r^2\mathrm{d}\varphi$ for $r > 0$, one has $J[\Phi] = r^2h^2\mathrm{d}\varphi$, which equals $\kappa_0^2\mathrm{d}\varphi$ wherever $rh = \kappa_0$.

(b) The kinetic contribution $S \supset -\int\frac{1}{2}\nabla_a\Phi^*\nabla^a\Phi\,\mathrm{dvol}_g$ varies, after an integration by parts, to $\frac{1}{2}\mathrm{Box}_g\,\Phi$ in E_{Φ^*}. The potential contributes $-(\partial V_\Phi/\partial|\Phi|^2)\Phi$. For the disformal dependence, $\delta S_\mathrm{o} = \frac{1}{2}\int\mathscr{I}^{ab}\delta\tilde{g}_{ab}\,\mathrm{dvol}_g$ and $\delta_\Phi\tilde{g}_{ab} = B(\delta J_a J_b + J_a\delta J_b)$, hence $\frac{1}{2}\mathscr{I}^{ab}\delta\tilde{g}_{ab} = B\,\mathscr{I}^{ab}J_a\delta J_b$ by symmetry of $\mathscr{I}$. Restricting to a variation $\delta\Phi^* = \eta^*$ with $\delta\Phi = 0$ gives $\delta J_b = \frac{1}{2\mathrm{i}}(\eta^*\nabla_b\Phi - \Phi\nabla_b\eta^*)$. Substituting and integrating the second summand by parts produces the two current terms in , each with coefficient $1/(2\mathrm{i}) = -\mathrm{i}/2 = -\frac{\mathrm{i}}{2}$. The Euler operator therefore contains

$-\frac{\mathrm{i}}{2}B\mathscr{I}^{ab}J_a\nabla_b\Phi - \frac{\mathrm{i}}{2}\nabla_b(B\mathscr{I}^{ab}J_a\Phi)$, which is . The switch variation is the standard real-scalar Euler operator together with $\partial V_\Phi/\partial s$ and the explicit $B'(s)$ contraction against $\mathscr{I}^{ab}J_aJ_b$.

(c) Because $J_a[\Phi]$ is built from scalar fields and covariant differentials with no metric-connection dependence in the one-form itself, $\delta_{\hat{g}}\hat{g}_{ab} = \delta g_{ab}$ at fixed (Φ, s), giving $T^{ab}_{\mathrm{o}} = \mathscr{I}^{ab}$ in the g-frame. The sum $S_{\mathrm{o}} + S_\Phi + S_s$ is a diffeomorphism-invariant local functional of $(g, \psi_{\mathrm{o}}, \mathscr{A}, \Phi, \Phi^*, s)$. Under any infinitesimal diffeomorphism $\xi \in C^\infty_c(M; TM)$, simultaneous Lie dragging of all fields gives $\delta_\xi(S_{\mathrm{o}} + S_\Phi + S_s) = 0$; when ψ_{o}, Φ, and s are on shell, only the metric variation $\frac{1}{2}(T^{ab}_{\mathrm{o}} + T^{ab}_\Phi + T^{ab}_s)(2\nabla_{(a}\xi_{b)})$ survives, which is .

Theorem (On-shell toroidal realisation by a two-invariant potential, or on a *z*-independent radial collar). *Let g be a smooth stationary unit-lapse metric of the form below, and assume the ordinary sector is empty in the sense* $\mathscr{I}^{ab} = 0$. *For a stationary profile* $\Phi = h(r^2, z^2)(x + \mathrm{i}y)$ *one has the exact identity* $\mathrm{Box}_g\,\Phi = \Delta_{\mathbb{R}^3}\Phi$. *Writing* $H(R, Z) := h(R, Z)$ *with* $R = r^2$ *and* $Z = z^2$,

$$\Delta_{\mathbb{R}^3}\Phi = \lambda(r^2, z^2)\,\Phi, \qquad \lambda := 8\partial_R H + 4r^2\partial_R^2 H + 2\partial_Z H + 4z^2\partial_Z^2 H,$$

and λ extends smoothly to all of $\mathbb{R}^3$, *including the axis and the complement of* $\mathrm{supp}\,(h)$. *The official version 2 single-invariant pullback of λ, of* $u := |\Phi|^2 = r^2h^2$, *and of* $\Delta_{\mathbb{R}^3}s$ *to a single toroidal invariant* $\sigma = (r - r_0)^2 + z^2$ *is not coordinate-consistent on* $\mathbb{R}^3$ *and is withdrawn (clause (c)). The following two realisations replace it. Both are prescribed-host bookkeeping on the declared vacuum subclass* $\mathscr{I}^{ab} = 0$. *They are not Einstein dynamics, not Host C, not a derived metric, and not a solution of the coupled Einstein–vortex system.*

1. *(Two-switch potential on* (r^2, z^2), *under integrability.) Extend the single-switch action of Definition by two smooth real scalars* s_1, s_2 *and a joint potential* $W(s_1, s_2)$, *with formal Lagrangian density*

$$\frac{1}{2}g^{ab}\nabla_a\Phi^*\nabla_b\Phi + V_\Phi(|\Phi|^2, s_1, s_2) + \sum_{i=1}^{2}\frac{F_i^2}{2}g^{ab}\nabla_a s_i\nabla_b s_i + W(s_1, s_2),$$

the Euler–Lagrange equations being understood in the sense of compactly supported variations. Let $h = h(r^2, z^2)$

be smooth and compactly supported in $\{r \geq r_{\text{in}} > 0\}$*, and take the smooth invariants* $s_1 = r^2$ *and* $s_2 = z^2$ *on* $\mathbb{R}^3$*. Then* $\Delta_{\mathbb{R}^3} s_1 = 4$ *and* $\Delta_{\mathbb{R}^3} s_2 = 2$ *identically, while* λ *and* $u = s_1 h^2$ *become functions of* (s_1, s_2)*. Define*

$$V_{\Phi}(|\Phi|^2, s_1, s_2) := \frac{1}{2}\lambda(s_1, s_2)\,|\Phi|^2.$$

If the closedness condition

$$\frac{\partial \lambda}{\partial s_1}\frac{\partial u}{\partial s_2} = \frac{\partial \lambda}{\partial s_2}\frac{\partial u}{\partial s_1}$$

holds, equivalently if λ *and* u *have the same level sets on the printed image, then there exists a smooth* $W(s_1, s_2)$ *such that* $E_{\Phi^*} = 0$ *and both switch Euler equations* $F_i^2 \Delta_{\mathbb{R}^3} s_i - \partial W/\partial s_i - \partial V_{\Phi}/\partial s_i = 0$ *hold pointwise on* $\mathbb{R}^4$*. The construction is not claimed for a fully general two-variable amplitude* $h(r^2, z^2)$ *without . The switch fields* $s_1 = r^2$ *and* $s_2 = z^2$ *are not compactly supported; the controller* Φ *and the metric deformation remain compactly supported.*

2. *(z-independent radial collar, unconditionally.) If* $h = h(r^2)$ *and* $s = r^2$*, then* $\lambda = 8H' + 4r^2H''$ *and* $u = r^2 h(r^2)^2$ *are functions of the single invariant* s*,* $\Delta_{\mathbb{R}^3} s = 4$*, and*

$$V_{\Phi}(|\Phi|^2, s) := \frac{1}{2}\lambda(s)\,|\Phi|^2, \qquad W'(s) := 4F_s^2 - \frac{1}{2}\lambda'(s)\,u(s)$$

are well-defined smooth functions of s*. Thus* Φ *and* s *solve* $E_{\Phi^*} = 0$ *and* $E_s = 0$ *pointwise on all of* $\mathbb{R}^4$*, with no further integrability condition. A compactly supported reparametrization* $s = \chi(r^2)$ *may be used only on a range where* χ *is strictly monotone and the right-hand side of the switch equation descends as a function of* s*; a compactly supported bump is typically non-monotone and is not used as a global coordinate for* W*.*

3. *(Withdrawn single-invariant pullback.) If* h *and a switch* s *are merely functions of* $\sigma = (r - r_0)^2 + z^2$*, then neither* $u = r^2 h(\sigma)^2$ *nor* λ *from is, in general, a function of* σ *alone, because* r^2 *and* z^2 *vary independently along level tori of* σ*. Moreover* $\Delta_{\mathbb{R}^3}\sigma = 4 + 2(1 - r_0/r)$ *is not a function of* σ*, so* $\Delta_{\mathbb{R}^3} s$ *is not a function of* σ *for a non-constant* $s = s(\sigma)$*. A compactly supported* $s(\sigma)$ *is typically non-monotone, and* $W'(s)$ *is not a function of* s *alone. The official version 2*

formulae that treated $\lambda_\sigma(s)$ and $u(\sigma(s))$ as functions of s are therefore withdrawn. The Euler coefficient i/2 *of Theorem is not withdrawn.*

Proof. On the unit-lapse metric , one has $\det(g_{\mu\nu}) = -1$ and $g^{tt} = -1 + |\omega|^2$, $g^{ti} = -\omega^i$, $g^{ij} = \delta^{ij}$. For a stationary scalar, $\mathrm{Box}_g \Phi = \partial_\mu(g^{\mu\nu}\partial_\nu\Phi)$ collapses to $\Delta_{\mathbb{R}^3}\Phi$, the mixed drag terms cancelling. The Cartesian computation $\Delta_{\mathbb{R}^3}(H \cdot (x + \mathrm{i}y))$ yields , so the vacuum Euler equation $E_{\Phi^*} = \frac{1}{2}\Delta_{\mathbb{R}^3}\Phi - (\partial V_\Phi/\partial|\Phi|^2)\Phi = 0$ holds as soon as $\partial V_\Phi/\partial|\Phi|^2 = \frac{1}{2}\lambda$.

(a) The choice enforces that identity once λ is a function of (s_1, s_2). For $s_1 = x^2 + y^2$ and $s_2 = z^2$ one has $\Delta_{\mathbb{R}^3}s_1 = \partial_{xx}s_1 + \partial_{yy}s_1 = 4$ and $\Delta_{\mathbb{R}^3}s_2 = \partial_{zz}s_2 = 2$. With $\mathscr{T}^{ab} = 0$, the two switch Euler equations become $\partial W/\partial s_1 = 4F_1^2 - \frac{1}{2}(\partial\lambda/\partial s_1)u$ and $\partial W/\partial s_2 = 2F_2^2 - \frac{1}{2}(\partial\lambda/\partial s_2)u$. For constant F_i, mixed derivatives of this gradient agree if and only if holds, in which case W exists locally by Poincaré quadrature on the printed image. The same diffeomorphism argument as in Theorem (c) gives on-shell conservation for compactly supported variations of the formal density .

(b) In the z-independent class, λ and u are functions of $R = r^2$ alone, $\Delta_{\mathbb{R}^3}s = 4$, and the single switch equation $F_s^2\Delta_{\mathbb{R}^3}s - W'(s) - \frac{1}{2}\lambda'(s)u(s) = 0$ is . The right-hand side is a function of $s = r^2$, which is a global smooth coordinate on $\{r > 0\}$ and extends smoothly through the axis. Outside $\mathrm{supp}(h)$ one has $\Phi = 0$ and $\Delta_{\mathbb{R}^3}\Phi = 0$, so both Euler equations continue to hold.

(c) Along a level torus of σ, the cylindrical radius r varies (unless the torus degenerates to a circle in the equatorial plane). Thus $u = r^2h(\sigma)^2$ is not constant on that torus. The coefficients r^2 and z^2 in likewise vary. In cylindrical coordinates, $\partial_r\sigma = 2(r - r_0)$ and $\partial_z\sigma = 2z$, so $\Delta_{\mathbb{R}^3}\sigma = 2 + (2/r)(r - r_0) + 2 = 4 + 2(1 - r_0/r)$, which depends on r and not only on σ. Hence $\Delta_{\mathbb{R}^3}s = s''(\sigma)|\nabla\sigma|^2 + s'(\sigma)\Delta_{\mathbb{R}^3}\sigma$ is not a function of σ for non-constant s. This withdraws the official version 2 inversion of $W'(s)$ on the single-invariant subclass.

Compactly supported toroidal dark chronology split with global Cauchy $\tilde{g}$

Theorem (Prescribed compactly supported toroidal chronology-split witness on $\mathbb{R}^4$). *On $M = \mathbb{R}^4$ with Cartesian coordinates (t, x, y, z), cylindrical radius $r = \sqrt{x^2 + y^2}$, and Minkowski metric $\eta = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, fix geometric radii $0 < r_{\mathrm{in}} < r_0 < r_{\mathrm{out}} < R_{\mathrm{ext}}$ and a vertical half-height $0 < z_{\mathrm{out}} < R_{\mathrm{ext}}$ such that $\mathscr{K}_{\mathrm{supp}} := \{(x, y, z) : r \le r_{\mathrm{out}}, |z| \le z_{\mathrm{out}}\} \subset B_{R_{\mathrm{ext}}}(0)$. Choose a smooth non-negative bump $w \in C_c^\infty(\mathbb{R}^3; [0, 1])$ depending only on (r^2, z^2) such that*

• $\mathrm{supp}\,(w) \subseteq \mathscr{K}_{\mathrm{supp}} \cap \{r \ge r_{\mathrm{in}}\}$ *(in particular $w \equiv 0$ on $\{r \le r_{\mathrm{in}}\}$ and outside $B_{R_{\mathrm{ext}}}(0)$);*

• *w attains its strict global maximum $w = 1$ along the toroidal core ring $\mathscr{R}_* := \{r = r_0,\ z = 0\}$, with $\partial_r w(r_0, 0) = \partial_z w(r_0, 0) = 0$. Fix a frame-dragging parameter $\Omega_0 > \frac{1}{r_0}$, and prescribe on $\mathbb{R}^4$ the stationary unit-lapse coframe metric*

$$g := -(\mathrm{d}t + \Omega_0 w(r^2, z^2)(x\,\mathrm{d}y - y\,\mathrm{d}x))^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

(This is the intended host from the first line of the statement; a Version 1 cross-term ansatz without the quadratic $(\Omega_0 w(x\,\mathrm{d}y - y\,\mathrm{d}x))^2$ contribution is not used.) Choose a smooth radial-vertical amplitude $h \in C_c^\infty(\mathbb{R}^3; [0, \infty))$ of (r^2, z^2) supported inside $B_{R_{\mathrm{ext}}}(0)$ such that $r\,h(r^2, z^2) = 1$ everywhere on $\mathrm{supp}\,(w)$ (possible since $\mathrm{supp}\,(w) \subset \{r \ge r_{\mathrm{in}} > 0\}$), set $\Phi(x, y, z) := h(r^2, z^2)(x + \mathrm{i}y) \in C_c^\infty(\mathbb{R}^3; \mathbb{C})$ (extended stationarily in t), and choose a constant disformal coupling

$$B_0 > \Omega_0^2 r_{\mathrm{out}}^4.$$

(The Version 1 threshold $B_0 > \Omega_0^2 r_{\mathrm{out}}^2$ is dimensionally insufficient: writing $\mu := \Omega_0 r^2 w$ one has $|\mu| \le \Omega_0 r_{\mathrm{out}}^2$, not $\Omega_0 r_{\mathrm{out}}$.) Then on $M = \mathbb{R}^4$ with $\tilde{g}_{ab} := g_{ab} + B_0 J_a[\Phi] J_b[\Phi]$:

1. *(Global $C^\infty(\mathbb{R}^4)$ regularity and exact Minkowskian exterior.) Both g and $\tilde{g}$ are C^∞ Lorentzian metrics on all of $\mathbb{R}^4$. The coframe $\Theta^0 = \mathrm{d}t + \Omega_0 w(x\,\mathrm{d}y - y\,\mathrm{d}x)$, $\Theta^i = \mathrm{d}x^i$ has Jacobian determinant 1 relative to $(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, hence*

$\det(g_{\mu\nu}) \equiv -1$ *everywhere on* $\mathbb{R}^4$. *Both metrics coincide identically with* η *outside* $B_{R_{\text{ext}}}(0)$ *and in a neighbourhood of the axis* $\{r \le r_{\text{in}}\}$, *and* $\tilde{g}$ *is smooth across* $r = 0$ *because* $J[\Phi] = O(r^2)$ *there. The Killing field* ∂_t *is g-timelike and* $\tilde{g}$*-timelike, with* $g(\partial_t, \partial_t) = \tilde{g}(\partial_t, \partial_t) = -1$.

2. *(Toroidal dark CTC collar and a closed dark timelike geodesic.) For* $r > 0$, *reads* $g = -\mathrm{d}t^2 - 2\Omega_0 r^2 w\,\mathrm{d}t\,\mathrm{d}\varphi + r^2(1 - \Omega_0^2 r^2 w^2)\,\mathrm{d}\varphi^2 + \mathrm{d}r^2 + \mathrm{d}z^2$. *Inside the open toroidal tube* $\mathscr{U}_{\text{CTC}} := \{(t,x,y,z) \in \mathbb{R}^4 : \Omega_0 r w(r^2, z^2) > 1\}$, *which is nonempty and contains* $\mathbb{R}_t \times \mathscr{R}_*$ *because* $\Omega_0 r_0 > 1$, *every azimuthal circle* $\gamma(\varphi) = (t_0, r\cos\varphi, r\sin\varphi, z)$ *is a smooth closed future-directed g-timelike curve. Moreover, the continuous function* $C_-(r,z) := \Omega_0^2 r^4 w(r^2, z^2)^2 - r^2$ *on the compact annulus* $\{r_{\text{in}} \le r \le r_{\text{out}}, |z| \le z_{\text{out}}\}$ *satisfies* $C_-(r_0, 0) = r_0^2(\Omega_0^2 r_0^2 - 1) > 0$ *and* $C_- = -r^2 < 0$ *on* $\partial\,\mathrm{supp}\,(w)$, *so a maximum is attained at an interior point* (r_*, z_*) *where* $g_{\varphi\varphi}(r_*, z_*) = -C_-(r_*, z_*) < 0$ *and* $\partial_r g_{\varphi\varphi} = \partial_z g_{\varphi\varphi} = 0$. *By Killing's identity* $\nabla_{\partial_\varphi}\partial_\varphi = -\frac{1}{2}\nabla g_{\varphi\varphi} = 0$, *every circle* $(t_0, r_*\cos\varphi, r_*\sin\varphi, z_*)$ *is a smooth, closed, future-directed g-timelike geodesic in* $\mathbb{R}^4$. *Existence of this geodesic is not manufacture and is not Bar 3.*

3. *(Global hyperbolicity of* $(\mathbb{R}^4, \tilde{g})$.*) Because* $rh = 1$ *on* $\mathrm{supp}\,(w)$, *so that* $J[\Phi] = \mathrm{d}\varphi$ *there, and because holds, one has*

$$\tilde{g}_{\varphi\varphi}(r,z) \ge r^2$$

everywhere on $\mathbb{R}^4 \setminus \{r = 0\}$. *Writing* $J := h^2(x\,\mathrm{d}y - y\,\mathrm{d}x)$ *and* $\omega_{\text{drag}} := \Omega_0 w(x\,\mathrm{d}y - y\,\mathrm{d}x)$, *the identity* $\omega_{\text{drag}} = \mu J$ *holds globally on* $\mathbb{R}^4$ *with* $\mu := \Omega_0 r^2 w \in C_c^\infty(\mathbb{R}^3)$ *and* $|\mu| \le \Omega_0 r_{\text{out}}^2$, *hence* $B_0 - \mu^2 \ge B_0 - \Omega_0^2 r_{\text{out}}^4 =: \delta_B > 0$. *Completing the square gives the global Cartesian identity*

$$\tilde{g} = -\left(\frac{B_0}{B_0 - \mu^2}\right)\mathrm{d}t^2 + (B_0 - \mu^2)\left(J - \frac{\mu}{B_0 - \mu^2}\,\mathrm{d}t\right)^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Along every future-directed $\tilde{g}$*-causal curve one therefore has* $\dot{t} > 0$ *and the uniform Euclidean speed bound*

$$\left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{\mathbb{R}^3}^2 \le \frac{B_0}{\delta_B} =: N_{\max}^2 < \infty.$$

Any future- or past-inextendible $\tilde{g}$-causal curve parametrized by t has unbounded time extent, every hyperplane $\Sigma_{t_0} = \{t = t_0\}$ is a smooth global Cauchy surface, and every causal diamond is compact. Thus $(\mathbb{R}^4, \tilde{g})$ is globally hyperbolic on all of $\mathbb{R}^4$.

Proof. (i) The Jacobian statement for the coframe is immediate, and $\det(g_{\mu\nu}) = \det(\eta)\det(\Theta)^2 = -1$. Compact support of *w* and *h* gives the Minkowskian exterior and axis neighbourhood, and $g(\partial_t, \partial_t) = -1$ by inspection of .

(ii) Along $\gamma(\varphi) = (t_0, r\cos\varphi, r\sin\varphi, z)$ one has $x\,\mathrm{d}y - y\,\mathrm{d}x = r^2\mathrm{d}\varphi$, so $\Theta^0(\partial_\varphi) = \Omega_0 r^2 w > 0$ on $\mathcal{U}_{\mathrm{CTC}}$ (future-directed) and $g(\partial_\varphi, \partial_\varphi) = r^2(1 - \Omega_0^2 r^2 w^2) = -C_-(r,z) < 0$ whenever $\Omega_0 r w > 1$. The compact-annulus maximum argument supplies an interior critical point of C_-, hence of $-g_{\varphi\varphi}$. Because ∂_φ is Killing,

$g(\nabla_{\partial_\varphi}\partial_\varphi, X) = -\frac{1}{2}X(g(\partial_\varphi, \partial_\varphi)) = \frac{1}{2}X(C_-) = 0$ for all *X*, so the circle is a geodesic.

(iii) Outside $\mathrm{supp}(w)$ one has $\tilde{g}_{\varphi\varphi} = r^2 + B_0 h^4 r^4 \geq r^2$. On $\mathrm{supp}(w)$, the normalization $rh = 1$ gives $\tilde{g}_{\varphi\varphi} = r^2 - \Omega_0^2 r^4 w^2 + B_0 \geq r^2 + B_0 - \Omega_0^2 r_{\mathrm{out}}^4 > r^2$ by . Thus holds with room to spare. The identity $\omega_{\mathrm{drag}} = \mu J$ holds on $\mathrm{supp}(w)$ because $rh = 1$ forces $J = \mathrm{d}\varphi$ and $\omega_{\mathrm{drag}} = \Omega_0 r^2 w\,\mathrm{d}\varphi$, and holds outside $\mathrm{supp}(w)$ because both sides vanish (or, equivalently, $\mu = 0$). Expanding recovers $\tilde{g} = g + B_0 J \otimes J$. The middle term is nonnegative, the coefficient of $\mathrm{d}t^2$ is bounded by $N_{\max}^2$, and $\tilde{g}$-causality therefore implies . An inextendible causal curve with finite *t*-endpoint would have an $N_{\max}$-Lipschitz spatial track in the complete Euclidean space $\mathbb{R}^3$, hence a limiting point, contradicting inextendibility. Every constant-*t* plane is therefore a Cauchy surface, and causal diamonds are closed and bounded in $[t(p), t(q)] \times \overline{B}_{N_{\max}(t(q) - t(p))}(\mathbf{x}(p))$, hence compact.

par *Non-claim.* Theorem is a prescribed bimetric TM-2 geometric witness. It is not laboratory manufacture, not Einstein dynamics, not Host C, and not a discharge of Bar 1, Bar 2, or Bar 3. par

Proposition (Linearised transverse Jacobi equation about the closed ring). *In the setting of Theorem (ii), write $a(r,z) := \Omega_0 r^2 w(r^2, z^2)$ so that $C_- = a^2 - r^2$ and $g_{\varphi\varphi} = -C_-$. Let (r_*, z_*) be an interior critical point of C_-, and let γ be the corresponding closed g-timelike geodesic with affine parameter $\lambda = \varphi$, so that $t = 0$ and $\varphi = 1$ along γ.*

Conserved energy and angular momentum are $E = \dot{t} + a\dot{\varphi}$ *and* $\ell = -a\dot{t} + g_{\varphi\varphi}\dot{\varphi}$*. For isoenergetic, iso-angular perturbations, the linearized* (r, z) *deviation* $(\delta r, \delta z)$ *obeys*

$$\ddot{\delta r} + \left(\frac{C_-}{a^2} + \frac{1}{2}\partial_r^2 C_-\right)_* \delta r + \frac{1}{2}(\partial_r \partial_z C_-)_* \delta z = 0,$$

$$\ddot{\delta z} + \frac{1}{2}(\partial_z \partial_r C_-)_* \delta r + \frac{1}{2}(\partial_z^2 C_-)_* \delta z = 0,$$

where dots are λ-derivatives and the energy-coupling correction C_-/a^2 *in the r-channel arises because* $\partial_r a_* = r_*/a_* \neq 0$*, while* $\partial_z a_* = 0$ *at any critical point of* C_- *in the CTC region* $a \neq 0$*. If the profile is even in z and* $z_* = 0$*, the mixed derivatives vanish and the channels decouple. In the z-channel at a strict local maximum of* C_- *one has* $\partial_z^2 C_- < 0$*, so is hyperbolic rather than oscillatory: the linearised z-mode is unstable. The r-channel frequency (or growth rate) is controlled by the competition between the positive coupling* C_-/a^2 *and the negative Hessian entry* $\frac{1}{2}\partial_r^2 C_-$*, and is not typestated as automatically real. In particular, the formulae* $\omega_r^2 = -\partial_r^2 g_{\varphi\varphi}/(2C_-) > 0$ *and* $\omega_z^2 = -\partial_z^2 g_{\varphi\varphi}/(2C_-) > 0$ *at a maximum of* C_- *are not identities of the linearised system and are not used. This is linearised orbital analysis in a printed class. It is not a KAM theorem, and not a finite-thickness invariant phase-space tube of trapped Vlasov matter.*

Proof. The geodesic Lagrangian is $L = \frac{1}{2}(-\dot{t}^2 - 2a\dot{t}\dot{\varphi} + g_{\varphi\varphi}\dot{\varphi}^2 + \dot{r}^2 + \dot{z}^2)$. The r and z Euler–Lagrange equations are

$$\ddot{r} = -(\partial_r a)\dot{t}\dot{\varphi} + \frac{1}{2}(\partial_r g_{\varphi\varphi})\dot{\varphi}^2,$$

$$\ddot{z} = -(\partial_z a)\dot{t}\dot{\varphi} + \frac{1}{2}(\partial_z g_{\varphi\varphi})\dot{\varphi}^2.$$

The background $\dot{t} = 0$, $\dot{\varphi} = 1$, $\nabla g_{\varphi\varphi} = 0$ is therefore a geodesic. Linearising and using $\partial_z a_* = 0$ (from $\partial_z C_- = 2a\partial_z a = 0$ and $a_* \neq 0$) together with $\nabla g_{\varphi\varphi} = 0$ leaves a residual coupling $(\partial_r a)_* \dot{\delta t}$ in the r-channel only. The first-order constraints $\delta E = 0$ and $\delta\ell = 0$ at the background values $E_* = a_*$ and $\ell_* = g_{\varphi\varphi,*}$ give $\dot{\delta\varphi} = -(a_*/C_{-*})\dot{\delta t}$ and then $\dot{\delta t} = (C_{-*}/r_*^2)(\partial_r a_*)\delta r$. The critical-point identity $\partial_r C_- = 2a\partial_r a - 2r = 0$ forces $\partial_r a_* = r_*/a_*$, and therefore

$(\partial_r a)_* \dot{\delta t} = (C_{-*}/a_*^2)\delta r$. Substituting $g_{\varphi\varphi} = -C_-$ produces and . The sign of $\frac{1}{2}\partial_z^2 C_-$ at a strict local maximum is negative, which is the unstable oscillator equation $\ddot{\delta z} - |v|\delta z = 0$. Evenness in z at $z_* = 0$ kills the mixed Hessian. No KAM or Vlasov-confinement argument is supplied.

Proposition (Linearly oscillatory cratered ring in a printed profile). *The typestate of Proposition at a strict local maximum of C_- is not withdrawn: the vertical Jacobi mode remains hyperbolic there. The following printed profile is a different class, in which C_- has a positive local minimum inside the CTC tube.*

Fix $r_* = 2$ and $\Omega_0 = 1$, and define the local model

$$C_{\mathrm{loc}}(r,z) := 5 + (r-2)^2 + z^2.$$

Let $\eta \in C_c^\infty(\mathbb{R}^3;[0,1])$ be even in z, supported in the compact annular box $\{1 \le r \le 3,\ |z| \le 1\}$, and identically 1 on the flat neighbourhood $\{(r-2)^2 + z^2 \le \frac{1}{4}\}$. Set

$$w(r,z) := \eta(r,z)\,\frac{\sqrt{C_{\mathrm{loc}}(r,z) + r^2}}{r^2}$$

for $r > 0$, and $w = 0$ along the axis (which does not meet the support). Then w is nonnegative, even in z, smooth on $\mathbb{R}^3$, and compactly supported away from the axis. On the flat neighbourhood one has $C_-(r,z) = C_{\mathrm{loc}}(r,z)$, the point $(r_*,0) = (2,0)$ is a strict local minimum of C_- with $C_-(2,0) = 5 > 0$, and $\nabla C_-(2,0) = 0$. In particular $\Omega_0 r_* w(r_*,0) = 3/2 > 1$, so the azimuthal circle at $(2,0)$ is a closed future-directed g-timelike geodesic of the printed unit-lapse host, and the official Jacobi system of Proposition applies. The mixed Hessian vanishes by evenness, and the official Jacobi coefficients are strictly positive:

$$\left(\frac{C_-}{a^2} + \frac{1}{2}\partial_r^2 C_-\right)_* = \frac{14}{9} > 0, \qquad \left(\frac{1}{2}\partial_z^2 C_-\right)_* = 1 > 0.$$

Thus both transverse Jacobi modes are linearly oscillatory, with frequencies $\omega_r = \sqrt{14}/3$ and $\omega_z = 1$ in the affine parameter $\lambda = \varphi$. This is a printed closed dark timelike geodesic with linearly oscillatory transverse Jacobi modes in a prescribed class. It is not nonlinear orbital stability, not a KAM theorem, not Vlasov confinement, not Bar 3,

and not a manufactured time machine.

Proof. On $\{\eta = 1\}$ one has $a = \Omega_0 r^2 w = \sqrt{C_{\mathrm{loc}} + r^2}$, hence $C_- = a^2 - r^2 = C_{\mathrm{loc}}$. The jet of C_{loc} at $(2, 0)$ is $C_* = 5$, $\nabla C_{\mathrm{loc}} = 0$, $\partial_r^2 C_{\mathrm{loc}} = 2$, $\partial_z^2 C_{\mathrm{loc}} = 2$, and $\partial_r \partial_z C_{\mathrm{loc}} = 0$. Then $a_*^2 = C_* + r_*^2 = 9$, so $C_*/a_*^2 + \frac{1}{2}\partial_r^2 C_- = 5/9 + 1 = 14/9$ and $\frac{1}{2}\partial_z^2 C_- = 1$. The official derivation of – uses only that (r_*, z_*) is a critical point of C_- in the CTC region $a \neq 0$, together with evenness to kill the mixed term, and therefore applies verbatim. Both coefficients in are positive, so each decoupled equation is a linear oscillator in λ. (By contrast, at a strict local maximum of C_- one has $\partial_z^2 C_- < 0$, and the z-channel remains hyperbolic, as already typestated in Proposition .) Existence of the geodesic is Killing's identity, as in Theorem (ii), and is not Bar 3.

Unit-lapse Einstein tensor and linear convected Helmholtz operator

Proposition (Orthonormal Einstein tensor of the stationary unit-lapse ansatz). *Let g be any smooth stationary metric on* $\mathbb{R}^4$ *of the coframe form*

$$g = -(\mathrm{d}t + \omega_i(\mathbf{x})\,\mathrm{d}x^i)^2 + \delta_{ij}\,\mathrm{d}x^i \mathrm{d}x^j,$$

with $\omega \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ *and* $\mathbf{B}_{\mathrm{gm}} := \nabla_{\mathbb{R}^3} \times \omega$. *In the orthonormal coframe* $\Theta^0 = \mathrm{d}t + \omega_i \mathrm{d}x^i$, $\Theta^i = \mathrm{d}x^i$, *the Einstein tensor has components*

$$G_{\hat{0}\hat{0}} = +\frac{3}{4}|\mathbf{B}_{\mathrm{gm}}|^2_{\mathbb{R}^3},$$

$$G_{\hat{0}j} = -\frac{1}{2}(\nabla_{\mathbb{R}^3} \times \mathbf{B}_{\mathrm{gm}})_j,$$

$$G_{\widehat{\mathrm{jmath}}\,\hat{k}} = \frac{1}{4}|\mathbf{B}_{\mathrm{gm}}|^2_{\mathbb{R}^3}\delta_{jk} - \frac{1}{2}B_{\mathrm{gm},j}B_{\mathrm{gm},k}.$$

In particular $G_{\hat{0}\hat{0}} \geq 0$ *everywhere, with strict positivity wherever* $\mathbf{B}_{\mathrm{gm}} \neq 0$. *(A Version 1 statement printed the opposite density* $-\frac{1}{4}|\mathbf{B}_{\mathrm{gm}}|^2 \leq 0$*; that sign is false for and is withdrawn.) For any unit spatial vector* $\hat{u} \in S^2$ *and the g -null vectors* $k_\pm := e_0 \pm \hat{u}$,

$$G_{ab}k_\pm^a k_\pm^b = |\mathbf{B}_{\mathrm{gm}}|^2 - \frac{1}{2}(\mathbf{B}_{\mathrm{gm}} \cdot \hat{u})^2 \mp (\nabla_{\mathbb{R}^3} \times \mathbf{B}_{\mathrm{gm}}) \cdot \hat{u}.$$

A necessary condition for the null energy condition to hold at a point, for all null vectors, is therefore $|\nabla_{\mathbb{R}^3} \times \mathbf{B}_{\mathrm{gm}}| \le |\mathbf{B}_{\mathrm{gm}}|^2$. *Consequently, whenever* $|\nabla_{\mathbb{R}^3} \times \mathbf{B}_{\mathrm{gm}}(x_*)| > |\mathbf{B}_{\mathrm{gm}}(x_*)|^2$ *at some point* x_*, *the null energy condition fails at* x_*. *This sufficient criterion is realized in the following classes.*

1. *(Small-amplitude scalings.) If* $\omega_0 \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ *satisfies* $\nabla \times \omega_0 \equiv 0$, *then for the scaled profile* $\omega = \lambda\omega_0$ *the inequality* $|\nabla \times \mathbf{B}_{\mathrm{gm}}| > |\mathbf{B}_{\mathrm{gm}}|^2$ *holds, throughout the open set where* $\nabla \times (\nabla \times \omega_0) \neq 0$, *for every sufficiently small* $|\lambda| > 0$.

2. *(Printed toroidal family, any amplitude.) For the printed drag* $\omega_i \mathrm{d}x^i = \Omega_0 w(r^2, z^2)(x\,\mathrm{d}y - y\,\mathrm{d}x)$ *of Theorem , and for any* C^∞ *bump w which is not identically zero and vanishes along an inward normal to a farthest support point in the equatorial plane* $z = 0$, *the scalar comparison lemma of the proof implies that* $|w'(s)|/w(s)^2$ *is unbounded as the support edge is approached. In the absence of an accidental vanishing of* $\nabla \times \mathbf{B}_{\mathrm{gm}}$ *faster than this rate, one has* $|\nabla \times \mathbf{B}_{\mathrm{gm}}| > |\mathbf{B}_{\mathrm{gm}}|^2$ *in a boundary collar, for every* Ω_0, *including values large enough to form a CTC tube. The nondegeneracy is verified for standard radial flattening profiles, including* $w = \exp(-1/s)_+$ *along the inward normal. The sign of* $G_{\hat{0}\hat{0}}$ *is typestated separately from the null energy condition: a nonnegative energy density does not prevent a null-energy violation in a compact-support curl shell. The proposition is an obstruction for the printed unit-lapse, compact-support class. It is not an Einstein solution, not Host C, and not a chronology-protection theorem beyond the printed ansatz.*

Proof. The coframe satisfies $\mathrm{d}\Theta^0 = \frac{1}{2}F_{ki}\Theta^k \wedge \Theta^i$ with $F_{ij} = \partial_i\omega_j - \partial_j\omega_i$ and $F_{ij} = \varepsilon_{ijk}B^k_{\mathrm{gm}}$, while $\mathrm{d}\Theta^i = 0$. The unique metric-compatible, torsion-free connection is $\Omega^0{}_i = \frac{1}{2}F_{ij}\Theta^j$ and $\Omega^i{}_j = \frac{1}{2}F_{ij}\Theta^0$. The curvature forms are $R^0{}_i = \mathrm{d}\Omega^0{}_i + \Omega^0{}_k \wedge \Omega^k{}_i$ and $R^i{}_j = \mathrm{d}\Omega^i{}_j + \Omega^i{}_0 \wedge \Omega^0{}_j$. Evaluating on the orthonormal frame and contracting to Ricci yields

$$R_{\hat{0}\hat{0}} = \frac{1}{2}|\mathbf{B}_{\mathrm{gm}}|^2,$$

$$R_{\hat{0}\hat{j}} = -\frac{1}{2}(\nabla_{\mathbb{R}^3} \times \mathbf{B}_{\mathrm{gm}})_j,$$

$$R_{\widehat{\text{jmath}}\,\hat{k}} = \frac{1}{2}(|\mathbf{B}_{\text{gm}}|^2\delta_{jk} - B_{\text{gm},j}B_{\text{gm},k}).$$

(The identity $\sum_p F_{pj}F_{pk} = |\mathbf{B}_{\text{gm}}|^2\delta_{jk} - B_{\text{gm},j}B_{\text{gm},k}$ and $|F|^2 = 2|\mathbf{B}_{\text{gm}}|^2$ are used here.) The scalar curvature is $R = -R_{\hat{0}\hat{0}} + \delta^{jk}R_{\widehat{\text{jmath}}\,\hat{k}} = \frac{1}{2}|\mathbf{B}_{\text{gm}}|^2$. The Einstein tensor $G_{\hat{a}\hat{b}} = R_{\hat{a}\hat{b}} - \frac{1}{2}\eta_{\hat{a}\hat{b}}R$ is then exactly –; in particular $G_{\hat{0}\hat{0}} = R_{\hat{0}\hat{0}} + \frac{1}{2}R = +\frac{3}{4}|\mathbf{B}_{\text{gm}}|^2$. Contracting on $k_{\pm} = e_0 \pm \hat{u}$ produces . The minimizing choice $\hat{u} \parallel \nabla \times \mathbf{B}_{\text{gm}}$ gives the lower bound $G(k,k) \le |\mathbf{B}_{\text{gm}}|^2 - |\nabla \times \mathbf{B}_{\text{gm}}|$, so a point with $|\nabla \times \mathbf{B}_{\text{gm}}| > |\mathbf{B}_{\text{gm}}|^2$ violates the null energy condition.

For (a), $\mathbf{B}_{\text{gm}}$ and $\nabla \times \mathbf{B}_{\text{gm}}$ scale linearly in λ while $|\mathbf{B}_{\text{gm}}|^2$ scales quadratically, so the sufficient criterion holds for small $|\lambda|$ wherever $\nabla \times (\nabla \times \omega_0) \neq 0$.

For (b), let p be a point of $\text{supp}\,(w)$ of maximal Euclidean radius in the plane $z = 0$, and let s be inward radial distance from p. The restriction $f(s) := w(\gamma(s))$ is smooth, vanishes at $s = 0$, and is not identically zero on $(0, s_0]$ if p sees the support immediately. If $|f'| \le Cf^2$ held on a neighbourhood of $s = 0$, the function $M(s) := \sup_{[0,s]}|f|$ would satisfy $M(s) \le CsM(s)^2$. For small s one has $M(s) < 1/(Cs)$, hence $M \equiv 0$, a contradiction. Thus $\limsup_{s \to 0^+}|f'(s)|/f(s)^2 = +\infty$. Along $z = 0$ the printed field has $\mathbf{B}_{\text{gm}} = \Omega_0(0, 0, 2w - r\partial_s w)$ up to the inward-normal convention recorded after , so $|\mathbf{B}_{\text{gm}}|$ is of order $|\Omega_0|(w + |\nabla w|)$, while $\nabla \times \mathbf{B}_{\text{gm}}$ is of order $|\Omega_0|$ times second derivatives of w. For a standard flat bump $f(s) = \exp(-1/s)$ one has $|f''|/(f + |f'|)^2 \to +\infty$ as $s \to 0^+$, and the same leading comparison holds at any fixed Ω_0. This is the declared nondegeneracy. No curvature identity is imported from Dual-Ledger-Chrono-DM-B.

Remark (Energy typestate of the printed torus). Because $(\mathbb{R}^4, g)$ in equals Minkowski space outside a bounded cylinder while containing closed g-timelike curves inside $\mathscr{U}_{\text{CTC}}$, the chronology-protection literature leads one to expect an energy-condition obstruction. The correct obstruction for the printed unit-lapse class is the null contraction , not a negative energy density: one has $G_{\hat{0}\hat{0}} = +\frac{3}{4}|\mathbf{B}_{\text{gm}}|^2 \ge 0$. A Version 1 remark that printed $G_{\hat{0}\hat{0}} = -\frac{1}{4}|\nabla \times \omega_{\text{drag}}|^2 \le 0$ is withdrawn. The comparison with cylindrical Gödel or Som–Raychaudhuri source matches recorded in official Dual-Ledger-Chrono-DM-B version 4 [3] [Source record; not a premise.] is documentary only, is not typestated here as a winding-inclusive residual-free Einstein match, and is not used as a

premise.

Proposition (Linear convected Helmholtz / magnetic Schrödinger reduction). *On the printed unit-lapse metric , one has* $\det(g_{\mu\nu}) = -1$ *and the exact principal-symbol identity*

$$g^{ab}\xi_a\xi_b = -\xi_t^2 + |\xi_{\text{spatial}} - \xi_t\omega|^2_{\mathbb{R}^3}.$$

For a time-harmonic dark scalar $\psi_{\text{d}}(t,\mathbf{x}) = e^{-\mathrm{i}\omega t}u_\omega(\mathbf{x})$*, the linear operator* $-\mathrm{Box}_g + m_{\text{d}}^2$ *reduces to the magnetic Schrödinger, or convected Helmholtz, operator on* $\mathbb{R}^3$

$$[-(\nabla_{\mathbb{R}^3} + \mathrm{i}\omega\omega(\mathbf{x}))^2 + (m_{\text{d}}^2 - \omega^2)]u_\omega = \hat{J}_{\text{d}}(\omega,\mathbf{x}),$$

in the convention $(-\mathrm{Box}_g + m_{\text{d}}^2)\psi_{\text{d}} = J_{\text{d}}$ *of . For real* ω *and real* $\omega \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$*, the magnetic Laplacian* $-(\nabla + \mathrm{i}\omega\omega)^2$ *is essentially self-adjoint and nonnegative on* $L^2(\mathbb{R}^3)$*, and is uniformly elliptic on all of* $\mathbb{R}^3$*: its principal symbol is* $|\xi|^2$*, with no coefficient singularity on the chronology horizon* $\{\Omega_0 rw = 1\}$ *of the printed torus. If* $|\omega| < m_{\text{d}}$*, the operator in is strictly positive-definite, with lower bound* $m_{\text{d}}^2 - \omega^2 > 0$*, and is boundedly invertible on* $L^2(\mathbb{R}^3)$*. If* $|\omega| > m_{\text{d}}$*, the operator is a short-range magnetic Helmholtz operator, to which the standard Agmon–Hörmander outgoing resolvent theory applies as a named comparison.*

This is a linear operator identity on the prescribed stationary unit-lapse class. It is not a repair of Proposition , not Einstein compatibility, not unique Lorentzian continuation across a chronology horizon, and not a nonlinear H^s *existence theorem on an* $\mathbb{R}^4$ *toroidal time machine. Compact support of* ω*, in the printed class, remains the unit-lapse class whose energy typestate was just stated. Nonlinear existence on* $(\mathbb{R}^4, g)$ *is left residual in Appendix* .

Proof. The coframe matrix relating (Θ^0,Θ^i) to $(\mathrm{d}t,\mathrm{d}x^i)$ is unipotent with determinant 1, so $\det(g_{\mu\nu}) = -1$. The dual frame is $e_0 = \partial_t$ and $e_i = \partial_i - \omega_i\partial_t$, which is . For $\psi = e^{-\mathrm{i}\omega t}u(\mathbf{x})$, a direct expansion of $\mathrm{Box}_g\psi = \partial_\mu(g^{\mu\nu}\partial_\nu\psi)$ using $g^{tt} = -1 + |\omega|^2$, $g^{ti} = -\omega^i$, and $g^{ij} = \delta^{ij}$ produces $\mathrm{Box}_g\psi = e^{-\mathrm{i}\omega t}[(\nabla + \mathrm{i}\omega\omega)^2 u + \omega^2 u]$, and therefore $(-\mathrm{Box}_g + m_{\text{d}}^2)\psi = e^{-\mathrm{i}\omega t}[-(\nabla + \mathrm{i}\omega\omega)^2 u + (m_{\text{d}}^2 - \omega^2)u]$, which is . Essential self-adjointness and nonnegativity of the

magnetic Laplacian for a smooth, real, compactly supported vector potential are standard, as is ellipticity of the principal symbol $|\xi|^2$ at every point of $\mathbb{R}^3$. The subthreshold bound is immediate from nonnegativity. The superthreshold statement is recorded as a comparison with the existing outgoing theory for short-range magnetic potentials, not as a self-contained microlocal construction.

Proposition (Small-data stationary cubic dark fields on the prescribed unit-lapse torus). *On the printed unit-lapse metric , the linear identity reduces, at vanishing frequency $\omega = 0$, to the Yukawa equation*

$$(-\Delta_{\mathbb{R}^3} + m_{\mathrm{d}}^2)u_0 = \hat{J}_{\mathrm{d}}(0, \mathbf{x}),$$

with vanishing magnetic (frame-dragging) potential. Let $s > 3/2$, so that $H^s(\mathbb{R}^3)$ is a Banach algebra with constant $C_s > 0$, and write $K := (-\Delta_{\mathbb{R}^3} + m_{\mathrm{d}}^2)^{-1}$ for the Yukawa isomorphism, a bounded map $H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)$ with $\|K\|_{H^s \to H^s} \le 1/m_{\mathrm{d}}^2$. Then, for every real source $J_{\mathrm{d}} \in H^s(\mathbb{R}^3)$ and every cubic coupling $\lambda_{\mathrm{d}} \in \mathbb{R}$ satisfying the small-data, or small-coupling, bound

$$|\lambda_{\mathrm{d}}| < \frac{m_{\mathrm{d}}^2}{24\, C_s^2 R_0^2}, \qquad R_0 := \frac{2}{m_{\mathrm{d}}^2}\|J_{\mathrm{d}}\|_{H^s(\mathbb{R}^3)},$$

the stationary cubic equation

$$(-\mathrm{Box}_g + m_{\mathrm{d}}^2)\psi_{\mathrm{d}} + \lambda_{\mathrm{d}}\psi_{\mathrm{d}}^3 = J_{\mathrm{d}}(\mathbf{x})$$

admits a unique time-independent real solution $\psi_{\mathrm{d}} = u \in \overline{\mathscr{B}}_{R_0} \subset H^s(\mathbb{R}^3)$. If $J_{\mathrm{d}} \in C_c^\infty(\mathbb{R}^3)$ and holds for some $s_0 > 3/2$, then $u \in C^\infty(\mathbb{R}^3)$ by elliptic bootstrapping.

This is a small-data, or small-coupling, contraction in a printed Sobolev class on the prescribed stationary unit-lapse host. It is not a uniqueness statement for arbitrary J_{d}, not time-periodic nonlinear H^s existence, not a discharge of Appendix .2, not a repair of Proposition , not Einstein matching, and not unique continuation across a chronology horizon. Official linear Proposition is not withdrawn. The damped and cavity-quantized programmes remain in Appendix .

Proof. The reduction is the case $\omega = 0$ of , or equivalently the stationary identity $\mathrm{Box}_g = \Delta_{\mathbb{R}^3}$ already used in Theorem . Thus is $u = K(J_{\mathrm{d}} - \lambda_{\mathrm{d}} u^3)$. The map $T(u) := K(J_{\mathrm{d}} - \lambda_{\mathrm{d}} u^3)$ sends the closed ball of radius R_0 into itself once holds, because $\|KJ_{\mathrm{d}}\|_{H^s} \le R_0/2$ and, on that ball, $\|u^3\|_{H^s} \le C_s^2 R_0^2 \|u\|_{H^s}$, so the cubic term contributes at most $R_0/2$. The same estimate gives the contraction constant $\le 1/2$ on the ball, since $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$ and $\|u^2 + uv + v^2\|_{H^s} \le 3C_s^2 R_0^2$. The Banach fixed-point theorem supplies a unique fixed point in the ball. If J_{d} is smooth and compactly supported, T maps H^s into H^{s+2} for every $s > 3/2$, and iteration yields $u \in C^\infty(\mathbb{R}^3)$.

First-order equatorial annular arrival delays

Proposition (First-order equatorial annular arrival delays). *On the printed compactly supported unit-lapse torus of Theorem , restrict to equatorial future-directed null geodesics of g or of $\widehat{g}$, and write $a(r) := \Omega_0 r^2 w(r^2, 0)$ and $B_{\mathrm{eff}}(r) := B_0 r^4 h(r^2, 0)^4$ for the equatorial amplitudes (thus $B_{\mathrm{eff}} = B_0$ wherever $rh = 1$). Let $b = \ell/\varepsilon$ be the conserved asymptotic impact parameter, and write $\Delta t[g](b)$ (resp.?$\Delta t[\widehat{g}](b)$) for the regularized two-way coordinate-time delay relative to Minkowski space, obtained by subtracting the divergent Minkowski flight time at a large radius R and sending $R \to \infty$. Both delays vanish for $|b| > R_{\mathrm{ext}}$. The exact finite-amplitude arrival integrals are the definite integrals and ; they are not typestated as elementary closed forms for a general C^∞ profile.*

At first order in the compact amplitudes (a, B_{eff}), the delays of equatorial g-null and $\widehat{g}$-null rays are

$$\Delta t^{(1)}_{\mathrm{drag}}(b) = 2b \int_{|b|}^{\infty} \frac{a'(r) - a(r)/r}{\sqrt{r^2 - b^2}}\, \mathrm{d}r,$$

$$\Delta t^{(1)}_{B_0}(b) = \int_{|b|}^{\infty} \frac{B_{\mathrm{eff}}(r)/r - B'_{\mathrm{eff}}(r)}{\sqrt{r^2 - b^2}}\, \mathrm{d}r,$$

with $\Delta t[g]^{(1)} = \Delta t^{(1)}_{\mathrm{drag}}$ and $\Delta t[\widehat{g}]^{(1)} = \Delta t^{(1)}_{\mathrm{drag}} + \Delta t^{(1)}_{B_0}$. The drag increment is odd in b; the disformal increment is even in b, and its sign is that of the Abel integral , not typestated as uniformly positive for every piercing impact parameter. In particular, if a is piecewise constant and equal to a_0 on $[0, r_{\mathrm{out}}]$, one has the elementary first-order evaluation $\Delta t^{(1)}_{\mathrm{drag}}(b) = -2a_0 \arccos(b/r_{\mathrm{out}}) - 2a_0 b/\sqrt{r_{\mathrm{out}}^2 - b^2}$ for $0 < b < r_{\mathrm{out}}$.

The Abel integral $F(b) = \int_{|b|}^{\infty} f(r)/\sqrt{r^2 - b^2}\,\mathrm{d}r$ *inverts, for f of compact support and* $\rho > 0$*, by*

$$\int_{\rho}^{\infty} f(r)\,\mathrm{d}r = \frac{2}{\pi}\int_{\rho}^{\infty} \frac{b\,F(b)}{\sqrt{b^2 - \rho^2}}\,\mathrm{d}b, \qquad f(\rho) = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{\infty} \frac{b\,F(b)}{\sqrt{b^2 - \rho^2}}\,\mathrm{d}b.$$

Thus $f_a := a' - a/r$ *is recovered from the odd delay, and* $a(r) = r\int_r^{\infty} (-f_a(s)/s)\,\mathrm{d}s$ *with* $a(\infty) = 0$*; likewise* $f_B := B_{\mathrm{eff}}/r - B'_{\mathrm{eff}}$ *is recovered from the even delay, and* $B_{\mathrm{eff}}(r) = r\int_r^{\infty} (f_B(s)/s)\,\mathrm{d}s$*. In this first-order equatorial class, an odd/even split of paired co-rotating and counter-rotating* $\tilde{g}$*-null delays, or a pairing of g-null and* $\tilde{g}$*-null delays at the same b, separates* (a, B_{eff})*, hence* $(w, B_0 h^4)$ *on the equatorial plane. This is a prescribed-host comparison at first-order Abel exactness. It does not break official Chrono-DM-A Sagnac typestate, does not replace a host-null guided-ring reconstruction, and does not replace official Chrono-DM-B one-sided Sagnac identifiability; those statements are documentary neighbours and are not used as premises. Ordinary-only data remain fibre-blind in that official sense: a first-order equatorial Abel inversion of arrival times is not an identification of a chronology-split winding, nor of a genuine host-cone datum beyond the printed linearised pair* (a, B_{eff})*. Exact finite-amplitude* C^{∞} *deflection angles are not computed.*

Proof. On the equatorial unit-lapse host, the conserved energy and angular momentum give $\dot{t} = \varepsilon - a\dot{\varphi}$ and $r^2\dot{\varphi} = \ell + a\varepsilon$ along g-null geodesics, together with the null identity $\dot{r}^2 + (\ell + a\varepsilon)^2/r^2 = \varepsilon^2$. Writing $b = \ell/\varepsilon$ (for $\varepsilon > 0$) produces the exact flight integrand

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1 - a(b + a)/r^2}{\sqrt{1 - (b + a)^2/r^2}},$$

with turning point $r_m = |b + a(r_m)|$. The corresponding $\tilde{g}$-null integrand, using $\tilde{g}_{\varphi\varphi} = r^2 - a^2 + B_{\mathrm{eff}}$, is

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1 - a(b + a)/(r^2 + B_{\mathrm{eff}})}{\sqrt{1 - (b + a)^2/(r^2 + B_{\mathrm{eff}})}},$$

with turning point $r_m^2 + B_{\mathrm{eff}}(r_m) = (b + a(r_m))^2$. Both integrands reduce to the Minkowski kernel $r/\sqrt{r^2 - b^2}$ outside the compact supports, so the regularized delays are well-defined and vanish for $|b| > R_{\mathrm{ext}}$.

To obtain , set $s^2 = r^2 - (b + a(r))^2$ along a g-null ray with $b > 0$, so $s(r_m) = 0$ and

$\mathrm{d}t = [r^2 - a(b + a)]/[r(r - (b + a)a')]\,\mathrm{d}s$. For $a \equiv 0$ this is $\mathrm{d}t = \mathrm{d}s$, and $s_\infty = \sqrt{R^2 - b^2}$ is unperturbed for compact a. Expanding the integrand at fixed s, with the background radius $r_0 = \sqrt{s^2 + b^2}$, yields the first-order density $-(b/r_0^2)a + (b/r_0)a'$, and changing variables $\mathrm{d}s = r_0\,\mathrm{d}r_0/\sqrt{r_0^2 - b^2}$ produces . The case $b < 0$ follows by the substitution $b \mapsto -b$ in the turning relation $r_m = |b + a(r_m)|$, and gives an odd increment. For the disformal increment at $a = 0$, the substitution $s^2 = r^2 + B_{\mathrm{eff}}(r) - b^2$ likewise reduces the Minkowski integrand to $\mathrm{d}s$, and the first-order density $B_{\mathrm{eff}}/(2r_0^2) - B_{\mathrm{eff}}'/(2r_0)$ produces ; the kernel depends on b only through $|b|$ and b^2, hence is even. Cross terms between a and B_{eff} are second order. The piecewise-constant evaluation follows from in the distributional sense $a' = a_0(\delta_{\mathrm{in}} - \delta_{\mathrm{out}})$, or equivalently by expanding the elementary antiderivative $\sqrt{r^2 - (b + a_0)^2} - a_0 \arctan\left(\sqrt{r^2 - (b + a_0)^2}/(b + a_0)\right)$ of the exact inner integrand.

The identity $\int_\rho^r b\,\mathrm{d}b/\sqrt{(r^2 - b^2)(b^2 - \rho^2)} = \pi/2$ converts $F(b) = \int_{|b|}^\infty f(r)/\sqrt{r^2 - b^2}\,\mathrm{d}r$ into $\int_\rho^\infty bF(b)/\sqrt{b^2 - \rho^2}\,\mathrm{d}b = (\pi/2)\int_\rho^\infty f$, which is . The reconstructed $f_a = a' - a/r$ equals $r(a/r)'$, hence $a(r) = r\int_r^\infty (-f_a(s)/s)\,\mathrm{d}s$ once $a(\infty) = 0$; likewise $f_B = B_{\mathrm{eff}}/r - B_{\mathrm{eff}}'$ equals $-r(B_{\mathrm{eff}}/r)'$, hence $B_{\mathrm{eff}}(r) = r\int_r^\infty (f_B(s)/s)\,\mathrm{d}s$. Odd and even parts of the first-order $\hat{g}$ delay (or a g versus $\check{g}$ difference at fixed b) therefore isolate the two equatorial amplitudes. No winding reconstruction, and no official Sagnac fibre identification, is claimed from ordinary-only data.

The exact flight integrands and are recorded for reference. For a general smooth compact profile they remain definite integrals, not elementary antiderivatives. Even in the named piecewise-constant B_{eff} comparison with $a = 0$, the exact disformal flight integral $\int \sqrt{r^2 + B_0}/\sqrt{r^2 + B_0 - b^2}\,\mathrm{d}r$ is elliptic, and is not claimed as an elementary closed form.

Master synthesis of official Dual-Ledger-Chrono-DM-C version 3

Theorem (Synthesis of official Dual-Ledger-Chrono-DM-C version 3). *Within the declared model classes printed in Sections –:*

1. *on the cylindrical CTC host* (M, g)*, no globally single-valued smooth time function exists, yet* $-\mathrm{Box}_g + m_{\mathrm{d}}^2$ *is strictly hyperbolic along helical covers, and on non-resonant periodic blocks with a uniform gap* $\mu_{\mathrm{gap}} > 0$ *(explicitly inhabited by half-integer squared masses at* $a = 2$*,* $h = 3$*) admits a unique* C^∞ *self-consistent nonlinear solution in* H^s *(*$s > 2$*) under (Theorems and). A measure-generic Diophantine bound with derivative loss is available for the linear resolvent; nonlinear existence with derivative loss is not claimed (Proposition). On the non-compact cylinder, subthreshold Yukawa invertibility and a named superthreshold limiting-absorption comparison are recorded (Proposition);*

2. *for a localized internal dark detector transported along a closed g-geodesic and coupled across a compact gate to a globally hyperbolic ordinary probe, kinematic expectation periodicity yields Deutsch's fixed-point rule, whereas coherent trace periodicity yields* $A_{\mathrm{loop}} = \frac{1}{d}\mathrm{Tr}_D U$ *(Theorem). An explicit bilinear Magnus integration supplies a closed postselected zero-locus in the traceless commuting involutory-dipole class (Theorem (a)–(b)). In that same class, Deutsch's map admits a continuous family of* μ_D*-commuting fixed points and dephases* ρ_S *at* $\tau_D = I_D/d$*, whereas the postselected evaluation is the identity channel whenever* $\cos\theta \neq 0$*; a second-order Magnus term is computed in a printed non-commuting dark Floquet class (Theorem (c)–(d)). Both statements remain finite-detector seam comparisons, not continuum derivations, and not chronology theorems;*

3. *on topologically trivial* $M = \mathbb{R}^4$*, the complex controller* $\Phi = h(r^2, z^2)(x + \mathrm{i}y)$ *couples smoothly through* $J[\Phi]$ *with an explicit Euler operator and a four-slot conservation identity, and admits an on-shell potential-and-switch realisation by a two-invariant potential on* (r^2, z^2) *under an integrability condition, or unconditionally on a z -independent radial collar (Theorems and). The official version 2 single-invariant pullback is withdrawn as not coordinate-consistent. The prescribed unit-lapse host , with the corrected coupling threshold , carries closed future-directed dark timelike geodesics inside* $B_{R_{\mathrm{ext}}}(0)$ *while* $(\mathbb{R}^4, \tilde{g})$ *is globally hyperbolic on all of* $\mathbb{R}^4$ *and identically Minkowskian outside that ball (Theorem). Linearised Jacobi analysis about the closed ring does not yield an automatically stable epicyclic tube at a maximum of* C_- *(Proposition), while a printed cratered profile has both official Jacobi coefficients positive (Proposition);*

4. *the orthonormal Einstein tensor of the printed unit-lapse ansatz has nonnegative energy density* $G^{\wedge\wedge}_{00} = +\frac{3}{4}|\mathbf{B}_{\mathrm{gm}}|^2$, *correcting the Version 1 statement of the opposite sign, while the null energy condition can fail in a compact-support curl shell (Proposition). A linear magnetic Schrödinger reduction holds as an operator identity on the same ansatz (Proposition). A small-data, or small-coupling, Yukawa contraction yields a unique stationary cubic solution in a printed Sobolev class, and is not typestated as time-periodic nonlinear existence or Einstein matching (Proposition). First-order equatorial arrival delays are Abel integrals of the drag and of the disformal amplitude, with an odd/even separation of* $(w, B_0 h^4)$ *at that exactness; ordinary-only data remain fibre-blind in the official Sagnac sense, and exact finite-amplitude* C^∞ *deflection angles are not claimed (Proposition). Unfinished structural programmes—a curved-core Ori-type tail, damped or cavity-quantized and Nash–Moser nonlinear existence on the* $\mathbb{R}^4$ *torus, and a continuum path-integral reduction of the two seam rules—are recorded in Appendix and are not official theorems. All refusal boundaries of the Standing Ledger remain unchanged, and the inherited operational time-machine bar stays* 0 *of* 3 *as the full three-part bar.*

Proof. Items (i)–(iv) collect the theorems and propositions cited in the statement, together with the typestate restrictions printed there and in the corresponding non-claims. Appendix is excluded from the official claim list by construction.

Official Version 4 retains the master synthesis of official Version 3 unmoved. The identities recorded in Theorem are not upgraded to a manufactured-core or operational-CTC theorem. The toroidal core remains prescribed, not manufactured, and is not an operational CTC.

appendix

Unfinished programmes, named comparisons, and withdrawn pullback

The following items were proposed as structural breakthroughs, or are withdrawn official version 2 identities recorded for the debugging record. They are not official residual-resolving flagships. None of them advances the operational time-machine bar, creates Host C, or withdraws an official obstruction of the printed unit-lapse class.

Withdrawn official version 2 single-invariant pullback. Official version 2 Theorem claimed that if h and s were functions of $\sigma = (r - r_0)^2 + z^2$, then λ, $u = r^2h^2$, and $\Delta_{\mathbb{R}^3}s$ were functions of σ alone, so that $W'(s)$ was a function of s alone. That pullback does not survive: r^2 varies along level tori of σ, $\Delta_{\mathbb{R}^3}\sigma = 4 + 2(1 - r_0/r)$ is not a function of σ, and a compactly supported $s(\sigma)$ is typically non-monotone. The withdrawn formulae are not used in the official spine. The repaired realisation is Theorem of the present sheet.

II.1. Curved-core Ori-type, or Weyl–Lewis–Papapetrou, torus. Proposition remains an official obstruction for the printed unit-lapse, compact-support class, once its Einstein tensor is correctly restated. A non-unit-lapse Weyl–Lewis–Papapetrou, or Ori-core, ansatz with an asymptotically flat multipole tail,

$$g = -e^{2\nu(r,z)}(\mathrm{d}t + \Omega_0 w(r^2, z^2)(x\,\mathrm{d}y - y\,\mathrm{d}x))^2 + e^{2\psi(r,z)}(\mathrm{d}r^2 + \mathrm{d}z^2) + e^{2\alpha(r,z)}r^2\mathrm{d}\varphi^2,$$

may be considered as a programme. Separating a local Ricci-flat core from a compact-support curl shell does not automatically cancel the residual of Proposition . The identities "$G_{ab} = 0$ outside a ball", "the null energy condition holds everywhere on $\mathbb{R}^4$", and "the ordinary sector is globally hyperbolic and asymptotically Minkowskian" are not proved here for any printed curved-core metric and source. The construction is left unfinished. It is not typestated as a manufactured dark time machine. A prescribed Einstein-compatible TM-2 witness is not Bar 3, and does not advance 0 of 3.

II.2. Damped or cavity-quantized nonlinear existence on the $\mathbb{R}^4$ torus. Proposition supplies a linear operator identity, and Proposition supplies a small-data stationary cubic contraction. Neither supplies a nonlinear H^s existence theorem for time-periodic fields of the undamped equation $(\mathrm{Box}_g - m_\mathrm{d}^2)\psi_\mathrm{d} = \lambda_\mathrm{d}\psi_\mathrm{d}^3 + J_\mathrm{d}$ on the printed torus $(\mathbb{R}^4, g)$, and neither uniquely continues Lorentzian data across the chronology horizon in the sense of a characteristic initial-value problem. An added damping $\gamma_\mathrm{d}\partial_t\psi_\mathrm{d}$ is not the official undamped PDE, and a large spatial torus $\mathbb{T}^3_L$ is not $\mathbb{R}^4$. Those modified equations or manifolds are not written out as complete local arguments here, and they are not typestated as discharging the original undamped problem. They would not repair the unit-lapse null-energy obstruction, and would not supply unique continuation or Einstein matching. Official

Theorem and Proposition are not withdrawn. The existence question for the original undamped, time-periodic problem on $(\mathbb{R}^4, g)$ remains residual.

II.3. Gaussian Schwinger–Keldysh, or influence-functional, comparison. A formal Gaussian comparison is immediate from linearity: if a detector S is coupled bilinearly to a linear dark field whose two-point function is the outgoing, or Feynman, Green function of Proposition , then integrating out the dark field produces a linear completely positive, or a postselected Kraus, effective channel on S, according to the choice of boundary condition, and cannot by itself generate Deutsch's state-dependent nonlinear fixed point. This observation is a named comparison, not a general no-go theorem that Deutsch's map cannot arise from every linear path-integral boundary condition, nor from any state-independent quadratic field action beyond the printed coupling. No complete Keldysh computation on $(\mathbb{R}^4, g)$ is supplied. Official version 2's formal Gaussian remark is not inflated. In the official spine, finite Deutsch and postselected maps remain axiomatic toy seam rules in the sense of Theorems and . Even a completed derivation would not be Bar 1, Bar 2, or Bar 3.

II.4. Nash–Moser implicit function theorem on $M?_{er}$, and residual exact lensing. Official Proposition remains a linear resolvent bound $H^s \to H^{s-\tau}$ for $\tau > 4$. It is not nonlinear $C^\infty(M_{per})$ existence. A Nash–Moser upgrade, with dyadic Fourier smoothing and tame estimates on $u \mapsto u^3$, is not written here and stays unfinished. Official Theorem (uniform-gap Banach contraction, inhabited family $\mu_{gap} \geq \frac{1}{2}$) is not withdrawn.

First-order equatorial annular arrival delays, at the Abel exactness of Proposition , now live in the official spine. Exact finite-amplitude C^∞ deflection angles, non-equatorial integrals, and a stronger than first-order inverse remain uncomputed, and are left residual here. That residual comparison is not typestated as breaking the official Chrono-DM-A Sagnac non-identification fibre, and it does not replace the host-null guided-ring reconstruction of that sheet, nor official Chrono-DM-B one-sided Sagnac identifiability. Ordinary-only data remain fibre-blind unless a genuine host-cone datum is supplied, in the official sense recorded with Proposition .

Hand-verification log

HC-C1 (Dual cone algebra). For $g = -\mathrm{d}t^2 - 2a\mathrm{d}t\mathrm{d}\varphi - h\mathrm{d}\varphi^2$, one has $\det g_{(t,\varphi)} = -(a^2 - h) = -\Delta_0$, $g^{tt} = h/\Delta_0$, and $h\xi_t^2 - 2a\xi_t\xi_\varphi + \xi_\varphi^2 = (\xi_\varphi - a\xi_t)^2 - \Delta_0\xi_t^2$.

HC-C2 (Integer multiplier witness). At $a = 2$, $h = 3$ ($\Delta_0 = 1$), $T_0 = L_\perp = 2\pi$, $P(m, n, k_x, k_z) = (n - 2m)^2 - m^2 + k_x^2 + k_z^2 + m_\mathrm{d}^2 \in \mathbb{Z} + m_\mathrm{d}^2$, so $m_\mathrm{d}^2 \in \mathbb{Z} + \frac{1}{2}$ implies $|P| \geq \frac{1}{2}$.

HC-C3 (Complex current axis cancellation). For $\Phi = h(r^2, z^2)(x + \mathrm{i}y)$, $(hx)\mathrm{d}(hy) - (hy)\mathrm{d}(hx) = h^2(x\mathrm{d}y - y\mathrm{d}x) \in \Omega^1(\mathbb{R}^4)$ with no $1/r$ singularity.

HC-C4 (Corrected disformal threshold). On $\mathrm{supp}\,(w)$ one has $\mu = \Omega_0 r^2 w$ and $|\mu| \leq \Omega_0 r_\mathrm{out}^2$, so $B_0 - \mu^2 > 0$ throughout $\mathbb{R}^4$ once $B_0 > \Omega_0^2 r_\mathrm{out}^4$. The Version 1 bound $B_0 > \Omega_0^2 r_\mathrm{out}^2$ is not sufficient.

HC-C5 (Unit-lapse Einstein tensor). The Cartan computation on gives $R_{\hat{0}\hat{0}} = \frac{1}{2}|\mathbf{B}_\mathrm{gm}|^2$, $R = \frac{1}{2}|\mathbf{B}_\mathrm{gm}|^2$, and $G_{\hat{0}\hat{0}} = +\frac{3}{4}|\mathbf{B}_\mathrm{gm}|^2$, together with . No identity was imported from Chrono-DM-B.

HC-C6 (Controlled-Hadamard SWAP partial trace). The matrix elements of $U = (P_0 \otimes I + P_1 \otimes \mathrm{H})\mathrm{SWAP}$ yield $\mathrm{Tr}_D\, U = |0\rangle\langle 0| + |1\rangle\langle -|$, correcting a Version 1 expansion to $|1\rangle\langle +|$.

HC-C7 (Withdrawn single-invariant pullback; repaired collar). For $\sigma = (r - r_0)^2 + z^2$ one has $\Delta_{\mathbb{R}^3}\sigma = 4 + 2(1 - r_0/r)$, which is not a function of σ, while $u = r^2 h(\sigma)^2$ varies along level tori of σ. For the radial invariants, $\Delta_{\mathbb{R}^3}(r^2) = 4$ and $\Delta_{\mathbb{R}^3}(z^2) = 2$.

HC-C8 (Cratered official Jacobi coefficients). At the printed jet $r_* = 2$, $\Omega_0 = 1$, $C_{*} = 5$, $\partial_r^2 C_{-} = 2$, $\partial_z^2 C_{-} = 2$, one has $a_*^2 = 9$ and $C_{-}/a^2 + \frac{1}{2}\partial_r^2 C_{-} = 14/9 > 0$, $\frac{1}{2}\partial_z^2 C_{-} = 1 > 0$.

HC-C9 (Deutsch commutant and postselected identity channel). For $U = \exp(-\mathrm{i}\theta\mu_S \otimes \mu_D)$ in the traceless involutory class, the off-diagonal unknown $\tau_\perp$ in Deutsch's equation satisfies $(s^2 + c^2 m^2)\tau_\perp = 0$, hence $\tau_\perp = 0$ whenever $\sin\theta \neq 0$. The postselected operator is $A_\mathrm{loop} = \cos\theta\, I_S$, which normalises to the identity channel for

$\cos\theta \neq 0$.

HC-C10 (Stationary Yukawa reduction). At $\omega = 0$, the magnetic potential $i\omega\omega$ in vanishes, and Box_g reduces to $\Delta_{\mathbb{R}^3}$ on stationary scalars of the printed unit-lapse class.

HC-C11 (First-order delay parity). The regularized substitution $s^2 = r^2 - (b + a)^2$ (resp.?$s^2 = r^2 + B_{\mathrm{eff}} - b^2$) reduces the Minkowski flight integrand to d*s*, and the first-order densities yield an odd drag increment and an even disformal increment .

Standing ledger, restated. Host C is not created. O.1 remains NOT_CLOSED. XV-B is not opened; XVI-N and Route B are not discharged, except the class-limited negative of t441 XVI-N-Univ-A. Gate 0 and ST4b remain NOT_SUPPLIED. No inherited characteristic or radiative quantum coefficient is admitted. External geometric time remains supplied. The inherited time-machine bar remains 0 of 3 as the full three-part bar: Bar 1 (internal register / state restoration), Bar 2 (consumed clock-name / operational record return without an external archive), and Bar 3 (spacetime worldline / event return) all remain undischarged. No Page curve, island, bounce, spacetime emergence, phenomenal identification, or consciousness–dark-matter identification is made. H3, A1–A3, and the phenomenal-theory gate are unused and remain NOT_CLOSED. Host A and Host B remain separate documentary backgrounds. The source is not companion-wired: wired=false; declaredPath remains null.

Data availability. No new observations, numerical likelihoods, or exclusion contours are reported. The analytical definitions and proofs are contained in this source. No compilation or machine verification is asserted. Informal environment and brace checks, if recorded, are not a TeX compile.

Documentary negative controls

Table records a one-page documentary ledger of OpenAI catalogue families and refused slogans. The entries are unverified external catalogue items, cited only as documentary source records [10], and are isolated to this section. They are not theorem premises, not scheduler-derived cones, not typestate of a Page curve, island, bounce, or de Sitter state, and not a discharge of the operational time-machine bar, which remains 0 of 3 for ordinary observers

with Enter instruments. No retained official identity in this sheet depends on these entries. Written dark-ledger identities, where retained, remain on the written dark ledger and the printed class; they are not chronology engines in nature and not uniqueness of dark matter.

Continuum self-consistency on a prescribed CTC quotient, the finite-detector Unruh–DeWitt seam bridge, the smooth compactly supported unit-lapse toroidal witness on $\mathbb{R}^4$, the cratered-ring Jacobi signs, and the corrected unit-lapse energy density of Proposition remain typestate of official Version 3, retained in official Version 4. They are not upgraded to a manufactured-core or operational-CTC theorem. The core remains prescribed, not manufactured. The null-energy condition can still fail in a curl shell.

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[1] Cheung, N. (2026). Dual-Ledger-Chrono-DM-C: Continuum Dark-Field Self-Consistency, Detector Seam Rules, and a Prescribed Toroidal Chronology Split on $\mathbb{R}^4$. *Enter Calculus Companion*, official version 3 (request t508u). Typographic parent of the present official Version 4; not a premise. *Source record; not a premise.*

[2] Cheung, N. (2026). Dual-Ledger-Chrono-DM-A: Winding, Chronology Asymmetry, and Operational Insulation on Prescribed Lorentzian Hosts. Exact Geometric Witnesses, Corrected No-Paradox Results, and Complete-Record Rejection Contracts. *Enter Calculus Companion*, official version 4 (request t506u). *Source record; not a premise.*

[3] Cheung, N. (2026). Dual-Ledger-Chrono-DM-B: Class-Limited Einstein Matches, a Closed Causal-Geodesic Obstruction, On-Shell Ordinary Shields, a Prescribed Misner Overlay, and One-Sided Sagnac Identifiability on Prescribed Hosts. *Enter Calculus Companion*, official version 4 (request t509u). Official version 4 is not typestated here as a winding-inclusive residual-free Einstein match. *Source record; not a premise.*

[4] Cheung, N. (2026). Dual-Ledger-Mobility-A: Mobility asymmetry between an ordinary slot and a current-free dark slot on a prescribed host: a conditional fluid-plus-controller class, its conservation ledger, and declared inverse problems. *Enter Calculus Companion*, official version 5 (request t495u). *Source record; not a premise.*

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[10] Documentary unverified external catalogue entries: Oai365-A (Family 365), Oai290-A (Family 290), Oai265-A/B, Oai276-A, Oai212-A, and refused Family 260. Unverified documentary negative controls only; isolated to Section and Table ; never premises. Family 365 is boundary-metric recovery, not a scheduler-derived cone or a time machine. Family 290 is not a decoder of an unread complement and cannot decode $r_{\mathrm{ord}} = 1$. Oai265-A/B is not a Page curve; Oai276-A is not a free clock; Oai212-A is not worldline return. Family 260 remains refused.